ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stanley@tiac.net *on the Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-884 ***Proposed by M. N. Deshpande, Aurangabad, India***

Find an integer k such that the expression

$$F_n^2 F_{n+2}^2 + kF_{n+1}^2 F_{n+2}^2 + F_{n+1}^2 F_{n+3}^2$$

is a constant independent of n.

B-885 ***Proposed by A. J. Stam, Winsum, The Netherlands***

For $n > 0$, evaluate

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{k}{2n-k} \binom{2n-k}{n} F_{k+1}.$$

B-886 ***Proposed by Peter J. Ferraro, Roselle Park, NJ***

For $n \geq 9$, show that

$$\left\lfloor \sqrt[4]{F_n} \right\rfloor = \left\lfloor \sqrt[4]{F_{n-4}} + \sqrt[4]{F_{n-8}} \right\rfloor.$$

B-887 ***Proposed by A. J. Stam, Winsum, The Netherlands***

Show that

$$\sum_{k=0}^{n} \binom{y-n-1-k}{n-k} F_{2k+1} = \sum_{k=0}^{n} \binom{y-n-2-k}{n-k} F_{2k+2} = \sum_{j=0}^{n} \binom{y-j}{j}.$$

B-888 ***Proposed by A. Arya, J. Fellingham, and D. Schroeder, Ohio State University, OH, and J. Glover, Carnegie Mellon University, PA***

For $n \geq 1$, let $A_n = [a_{i,j}]$ denote the symmetric matrix with $a_{i,i} = i+1$ and $a_{i,j} = \min[i, j]$ for all integers i and j with $i \neq j$.

(a) Find the determinant of A_n.

(b) Find the inverse of A_n.

SOLUTIONS

nth Derivative

B-865 ***Proposed by Alexandru Lupas, University Lucian Blaga, Sibiu, Romania (Vol. 36, no. 5, November 1998)***

Let $f(x) = (x^2+4)^{n-1/2}$, where n is a positive integer. Let

$$g(x) = \frac{d^n f(x)}{dx^n}.$$

Express $g(1)$ in terms of Fibonacci and/or Lucas numbers.

Solution by Richard André-Jeannin, Cosnes et Romain, France

It is known (Theorem 2 from [1]) that

$$L_n(x) = 2\frac{n!}{(2n)!}\sqrt{x^2+4}\, g(x).$$

From this, we get

$$g(1) = \frac{(2n)!\,L_n}{2\sqrt{5}\,n!}.$$

Reference

1. Richard André-Jeannin. "Differential Properties of a General Class of Polynomials." *The Fibonacci Quarterly* **33.5** (1995):453-458.

Solutions also received by Paul S. Bruckman, H.-J. Seiffert, and the proposer.

Divisibility by 25

B-866 ***Proposed by the editor (Vol. 37, no. 1, February 1999)***

For n an integer, show that $L_{8n+4} + L_{12n+6}$ is always divisible by 25.

Solution 1 by Pentti Haukkanen, University of Tampere, Tampere, Finland

It is known [1, (17b)] that

$$L_{n+m} - (-1)^m L_{n-m} = 5F_nF_m.$$

Therefore,

$$L_{12n+6} + L_{8n+4} = 5F_{10n+5}F_{2n+1}.$$

It is well known that $a|b \Rightarrow F_a|F_b$. Therefore, $5|F_{10n+5}$ and, further,

$$25|5F_{10n+5}F_{2n+1} \text{ or } 25|L_{12n+6} + L_{8n+4}.$$

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section.* Chichester: Ellis Horwood Ltd., 1989.

Solution 2 by Calvin T. Long, Northern Arizona University, Flagstaff, AZ

More generally, we show that $L_r + L_{r+2s}$ is divisible by 25 if and only if $5|s$ or $5|r+s$. If we then take $r = 8n+4$ and $s = 2n+1$, we have $r+s = 10n+5$ and the above result follows.

It is well known (see, e.g., [1], p. 222) that

$$L_r + L_{r+2s} = \begin{cases} 5F_sF_{r+s} & \text{for } s \text{ odd,} \\ L_sL_{r+s} & \text{for } s \text{ even.} \end{cases}$$

Since $5 \nmid L_n$ for any n and $5|F_n$ if and only if $5|n$, it follows that $25|L_r + L_{r+s}$ if and only if $5|s$ or $5|r+s$.

Reference

1. C. Long. "On a Fibonacci Arithmetical Trick." *The Fibonacci Quarterly* **23.3** (1985):221-31.

Seiffert showed that $L_{2k} \equiv (-1)^{k-1}(5k^2-2) \pmod{25}$.

Solutions also received by Richard André-Jeannin, Brian D. Beasley, Paul S. Bruckman, Kathleen E. Lewis, Steve Scarborough, H.-J. Seiffert, Indulis Strazdins, and the proposer.

1999 Belongs

B-867 ***Proposed by the editor***
(Vol. 37, no. 1, February 1999)

Find small positive integers a and b so that 1999 is a member of the sequence $\langle u_n \rangle$, defined by $u_0 = 0, u_1 = 1, u_n = au_{n-1} + bu_{n-2}$ for $n > 1$.

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC

Since $u_2 = a$ and $u_3 = a^2 + b$, we may find a and b so that $a^2 + b = 1999$. The solution in positive integers that yields the largest a and hence the smallest b is $(a, b) = (44, 63)$. Such solutions range from $(44, 63)$ and $(43, 150)$ to $(1, 1998)$.

We note that since 1999 is prime and $u_4 = a(a^2 + 2b)$, the only way to achieve $u_4 = 1999$ is to take $(a, b) = (1, 999)$. Also, since $u_5 = a^4 + 3a^2b + b^2$, achieving $u_5 = 1999$ would force $a^4 < 1999$ or $a \in \{1, 2, 3, 4, 5, 6\}$, none of which produces an integer value for b.

Solutions also received by Indulis Strazdins and the proposer.

Congruence Mod 25

B-868 ***Based on a proposal by Richard André-Jeannin, Longwy, France***
(Vol. 37, no. 1, February 1999)

Find an integer $a > 1$ such that, for all integers n, $F_{an} \equiv aF_n \pmod{25}$.

Solution 1 by Pentti Haukkanen, University of Tampere, Finland

Note that $F_{25} = 75025$ is divisible by 25. By the well-known property $c|b \Rightarrow F_c|F_b$, we have $25|(F_{an} - aF_n)$ when $a = 25k$.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

We use induction to show that $a = 9$ is a solution.

For $n = 0$, $F_0 = 9F_0 = 0$. For $n = 1$, $F_9 = 34 \equiv 9 = 9F_1$ (mod 25). Given $n \geq 2$, we assume $F_{9(n-1)} \equiv 9F_{n-1}$ (mod 25) and $F_{9(n-2)} \equiv 9F_{n-2}$ (mod 25). For n an integer, it is straightforward to verify the identity $F_n = 76F_{n-9} + F_{n-18}$. Then

$$\begin{aligned} F_{9n} &= 76F_{9(n-1)} + F_{9(n-2)} \equiv 76(9F_{n-1}) + 9F_{n-2} \pmod{25} \\ &= 675F_{n-1} + 9(F_{n-1} + F_{n-2}) \equiv 9F_n \pmod{25}. \end{aligned}$$

Solution 3 (and generalization) by Richard André-Jeannin, Longwy, France

We shall prove that $F_{an} \equiv aF_n$ (mod 25) for all integers n if and only if $a \equiv 0$ (mod 25) or $a \equiv r$ (mod 20), where $r \in \{1, 5, 9\}$.

First, if $a \equiv 0$ (mod 25), it is well known that $F_{an} \equiv 0$ (mod 25) and the two members are divisible by 25.

Assuming now that a is not divisible by 25, and putting $n = 1$ and $n = 2$ in the relation $F_{an} \equiv aF_n$ (mod 25), we get that

$$\begin{cases} F_a \equiv a \pmod{25}, \\ F_a L_a = F_{2a} \equiv aF_2 = a \equiv F_a \pmod{25}. \end{cases}$$

From the last relation, we get that $F_a(L_a - 1) \equiv 0$ (mod 25), and thus that $L_a \equiv 1$ (mod 5) (recall that F_a is divisible by 25 only if a is divisible by 25). It is not hard to prove that the last relation holds if and only if $a \equiv 1$ (mod 4) or, equivalently, if and only if $a = 20k + r$, where $r \in \{1, 5, 9, 13, 17\}$.

We now need the following lemma.

Lemma: $F_{20k+r} \equiv 20k + r$ (mod 25) only if $r \in \{1, 5, 9\}$.

Proof: The sequence $X_k = F_{20k+r}$ satisfies the recurrence relation

$$X_k = L_{20}X_{k-1} - (-1)^{20}X_{k-2} = 15127X_{k-1} - X_{k-2} \equiv 2X_{k-1} - X_{k-2} \pmod{25}.$$

Any sequence of the form $(ck + d)$ is another solution of the recurrence

$$X_k = 2X_{k-1} - X_{k-2}.$$

From this, we see that, for every integer k, $F_{20k+r} \equiv (F_{20+r} - F_r)k + F_r$ (mod 25), since the two members satisfy the same recurrence and take the same value for $k = 0$ and for $k = 1$. Thus, we have to see that $F_{20+r} - F_r \equiv 20$ (mod 25) and that $F_r \equiv r$ (mod 25).

It is readily proven that 4 is the period (mod 25) of the sequence $Z_r = F_{20+r} - F_r$ and that $Z_r \equiv 20$ (mod 25) if and only if $r \equiv 1$ (mod 4) and, particularly, for $r \in \{1, 5, 9, 13, 17\}$. On the other hand, we have $F_r \equiv r$ (mod 25) for $r = 1, 5, 9$ when $F_{13} \equiv 8$ (mod 25) and $F_{17} \equiv 22$ (mod 25). This concludes the proof of the lemma.

Now, we have to distinguish two cases. Assuming first that $r = 1$ or $r = 9$, we see that 20 is the period of the sequence L_n (mod 25) and that $L_{20k+r} \equiv 1$ (mod 25) for $r = 1$ and $r = 9$. Now the sequence $Y_n = F_{(20k+r)n}$ satisfies the recurrence

$$Y_n = L_{20k+r}Y_{n-1} - (-1)^{20k+r}Y_{n-2} \equiv Y_{n-1} + Y_{n-2} \pmod{25},$$

since r is odd. Thus, the two sequences Y_n and $(20k+r)F_n$ satisfy the same recurrence and they take the same value (mod 25) for $n=0$ and for $n=1$ (by the lemma). We deduce from this that $F_{(20k+r)n} \equiv (20k+r)F_n$ (mod 25) for every integer n.

Finally, assuming that $r=5$, we see that $L_{20k+5} \equiv 1$ (mod 5) [since $20k+5 \equiv 1$ (mod 4)] and that the sequence $U_n = F_{(20k+5)n}/5$ is a sequence of integers, since 5 divides $20k+5$. Now, the sequence U_n satisfies the recurrence

$$U_n = L_{20k+5}U_{n-1} - (-1)^{20k+5}U_{n-2} \equiv U_{n-1} + U_{n-2} \pmod 5.$$

Thus, the two sequences U_n and $(4k+1)F_n$ satisfy the same recurrence mod (5), and they take the same values (mod 5) for $n=0$ and $n=1$, since, by the lemma, we can write $F_{20k+5}/5 \equiv 4k+1$ (mod 5). We deduce from this that, for every integer n, $F_{(20k+5)n}/5 \equiv (4k+1)F_n$ (mod 5) and thus that $F_{(20k+5)n} \equiv (20k+5)F_n$ (mod 25). This concludes the proof.

Another Generalization of B-868

Consider the general recurrence $W_n = PW_{n-1} - QW_{n-2}$ with the solutions U_n $(U_0 = 0,\ U_1 = 1)$ and V_n $(V_0 = 2, V_1 = P)$. The sequence of integers

$$X_n = \frac{U_{pn}}{U_p}$$

satisfies the recurrence

$$X_n = V_p X_{n-1} - Q^p X_{n-2}.$$

If p is an odd prime, it is well known that $V_p \equiv P \pmod p$ and that $Q^p \equiv Q \pmod p$. From this, we see that

$$X_n \equiv PX_{n-1} - QX_{n-2} \pmod p.$$

Reasoning as in the solution of the problem, we get that, for every integer n:

$$\frac{U_{pn}}{U_p} \equiv U_n \pmod p \quad \text{or that} \quad U_{pn} \equiv U_p U_n \pmod{pU_p}.$$

Solutions also received by H.-J. Seiffert and Indulis Strazdins.

A Polynomial for F

B-869 ***Based on a communication by Larry Taylor, Rego Park, NY (Vol. 37, no. 1, February 1999)***

Find a polynomial $f(x)$ such that, for all integers n, $2^n F_n \equiv f(n)$ (mod 5).

Solution by Indulis Strazdins, Riga Technical University, Riga, Latvia

For $n = 0, 1, 2, 3, 4$ (mod 5), the period of $2^n F_n$ is $(0, 2, 4, 1, 3)$, which coincides with the period of $2n$. Hence, $f(x) = (5m+2)x$ for any integer m.

Seiffert showed that, for n a nonnegative integer,

$$2^n F_n \equiv \frac{n}{3}(5n^2 - 15n + 16) \pmod{50}.$$

Solutions also received by Richard André-Jeannin, Brian D. Beasley, Don Redmond, H.-J. Seiffert, and the proposer.

Trigonometric Diophantine Equation

B-870 ***Proposed by Richard André-Jeannin, Longwy, France***
(Vol. 37, no. 1, February 1999)

Solve the equation

$$\tan^{-1} y - \tan^{-1} x = \tan^{-1} \frac{1}{x+y}$$

in nonnegative integers x and y, expressing your answer in terms of Fibonacci and/or Lucas numbers.

Solution by the proposer

Let $\theta = \tan^{-1} y - \tan^{-1} x$. It is clear that $-\pi/2 < \theta < \pi/2$, since x and y are nonnegative. Thus, the original equation is equivalent to

$$\frac{1}{x+y} = \tan\theta = \frac{y-x}{1+xy},$$

which can be written as $y^2 - x^2 = 1 + xy$ or

$$(2y-x)^2 - 5x^2 = 4. \tag{1}$$

It is well known that the nonnegative solutions of the Diophantine equation $Y^2 - 5X^2 = 4$ are given by $X = F_{2n}$ and $Y = L_{2n}$. From this, we see that the solutions of (1) are given by $x = F_{2n}$ and $y = (F_{2n} + L_{2n})/2 = F_{2n+1}$.

Solutions also received by Charles K. Cook, H.-J. Seiffert, and Indulis Strazdins.

Errata: In the solution to problem B-864 (August 1999), in the line after display (2), insert "and $n = j$" at the end of the line. In the next display after display (2), "since $Q_a = 1$" should read "since $Q_a \equiv 1$".

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