NETWORK PROPERTIES OF A PAIR OF GENERALIZED POLYNOMIALS

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1. INTRODUCTION

Ladder networks have been studied extensively using Fibonacci numbers, Chebyshev polynomials, Morgan-Voyce polynomials, Jacobsthal polynomials, etc. ([10], [11], [2], [14], [9]. [5], [3], and [4]). All these polynomials are, in fact, particular cases of the generalized polynomials defined by

$$U_n(x, y) = xU_{n-1}(x, y) + yU_{n-2}(x, y), \quad (n \geq 2) \tag{1a}$$

with

$$U_0(x, y) = 0, \quad U_1(x, y) = 1, \tag{1b}$$

and

$$V_n(x, y) = xV_{n-1}(x, y) + yV_{n-2}(x, y), \quad (n \geq 2) \tag{2a}$$

with

$$V_0(x, y) = 2, \quad V_1(x, y) = x. \tag{2b}$$

We first show that rational functions derived from the ratios of these polynomials may in fact be synthesized using two-element-kind electrical networks. As particular cases, we will show that the networks realized using Fibonacci and Lucas polynomials, or Pell and Pell-Lucas polynomials are reactance of LC-networks, while those using Jacobsthal polynomials are RC or RL networks. Based on these results, we will establish some elegant relations among the various polynomials, as well as some results regarding the location of the zeros of these polynomials, and also their derivative polynomials. One of the results we need for our development is the following:

$$V_n(x, y) = U_{n+1}(x, y) + yU_{n-1}(x, y) = xU_n(x, y) + 2yU_{n-1}(x, y), \quad (n \geq 1), \tag{3}$$

which can be established easily by induction. We may also show that $U_{2n}(x, y)$ is an odd polynomial in x of degree $(2n-1)$ and a polynomial in y of degree $(n-1)$, while $U_{2n+1}(x, y)$ is an even polynomial in x of degree $2n$ and a polynomial in y of degree n. Further, $V_{2n}(x, y)$ is an even polynomial in x of degree $2n$ and a polynomial in y of degree n, while $V_{2n+1}(x, y)$ is an odd polynomial in x of degree $(2n+1)$ and a polynomial in y of degree n.

2. SYNTHESIS WITH $U_n(x, y)$ AND $V_n(x, y)$

Consider the function $\frac{U_{2n+1}(x,y)}{U_{2n}(x,y)}$; we will express this as a continued fraction.

$$\frac{U_{2n+1}(x, y)}{U_{2n}(x, y)} = \frac{xU_{2n}(x, y) + yU_{2n-1}(x, y)}{U_{2n}(x, y)}$$

$$= x + \cfrac{1}{\cfrac{U_{2n}(x, y)}{yU_{2n-1}(x, y)}} = x + \cfrac{1}{\cfrac{xU_{2n-1}(x, y) + yU_{2n-2}(x, y)}{yU_{2n-1}(x, y)}}$$

$$= x + \cfrac{1}{\cfrac{x}{y} + \cfrac{1}{\cfrac{U_{2n-1}(x,y)}{U_{2n-2}(x,y)}}} = \cdots = x + \cfrac{1}{\cfrac{x}{y} + \cfrac{1}{x + \cfrac{1}{\cfrac{x}{y} + \cdots + \cfrac{1}{x + \cfrac{1}{\cfrac{x}{y}}}}}}. \tag{4}$$

If we now consider $\frac{U_{2n+1}(x,y)}{U_{2n}(x,y)}$ as the driving point impedance (DPI) of a one-port network consisting of two kinds of elements, whose impedances are proportional to x and (y/x), then the function $\frac{U_{2n+1}(x,y)}{U_{2n}(x,y)}$ given by (4) may be realized by the network of Fig. 1(a), where there are n elements whose impedances are proportional to x, and n other elements whose impedances are proportional to (y/x). It is observed that, if y equals a positive constant, say α, and $x = s$ (the complex frequency variable), then the element x corresponds to an inductor of value 1H, while the element (y/x) corresponds to a capacitor of value $(1/\alpha)$F. On the other hand, if $y = s$ and x is a positive constant, say β, then they correspond, respectively, to a resistor of β Ohms and an inductor of value$(1/\beta)$H.

We may similarly express

$$\frac{U_{2n}(x,y)}{U_{2n-1}(x,y)}, \quad \frac{V_{2n+1}(x,y)}{V_{2n}(x,y)}, \quad \text{and} \quad \frac{V_{2n}(x,y)}{V_{2n-1}(x,y)}$$

by continued fractions, and realize them as the DPIs of the one-ports shown in Figs. 1(b), 1(c), and 1(d), respectively. Now let us synthesize $\frac{V_{2n}(x,y)}{U_{2n}(x,y)}$ as the DPI of a ladder network. We have, from (3),

$$Z_{in} = \frac{V_{2n}(x,y)}{U_{2n}(x,y)} = \frac{xU_{2n}(x,y) + 2yU_{2n-1}(x,y)}{U_{2n}(x,y)}$$

$$= x + \cfrac{1}{\cfrac{U_{2n}(x,y)}{2yU_{2n-1}(x,y)}} = x + \cfrac{1}{\cfrac{xU_{2n-1}(x,y) + yU_{2n-2}(x,y)}{2yU_{2n-1}(x,y)}} = x + \cfrac{1}{\cfrac{x}{2y} + \cfrac{1}{2\cfrac{U_{2n-1}(x,y)}{U_{2n-2}(x,y)}}}. \tag{5}$$

It is observed that $2\frac{U_{2n-1}(x,y)}{U_{2n-2}(x,y)}$ is an impedance and may be realized by the network of Fig. 1(a), where all the impedances are now scaled by a factor of 2. Thus, $\frac{V_{2n}(x,y)}{U_{2n}(x,y)}$ may be realized as the DPI of the ladder network shown in Fig. 1(e). Similarly, $\frac{V_{2n+1}(x,y)}{U_{2n+1}(x,y)}$ may be realized as the DPI of the two-element-kind network of Fig. 1(f).

3. FIBONACCI, LUCAS, PELL, AND PELL-LUCAS POLYNOMIALS AND LADDER NETWORKS

Let us first consider the case when $x = s$ and $y = \alpha$, a positive constant; that is, we are dealing with $U_n(s,\alpha)$ and $V_n(s,\alpha)$. When $\alpha = 1$, they reduce to the Fibonacci and Lucas polynomials $F_n(s)$ and $L_n(s)$, respectively. Hence, we shall call $U_n(s,\alpha)$ and $V_n(s,\alpha)$ modified Fibonacci and

Lucas polynomials, and denote them by $\tilde{F}_n(s)$ and $\tilde{L}_n(s)$, respectively. It is then evident from the results of the previous section that $\tilde{F}_{2n+1}(s)/\tilde{F}_{2n}(s)$ may be realized as the DPI of the reactance network given by Fig. 1(a), where each of the series elements corresponds to an inductor of value 1H and each of the shunt elements corresponds to a capacitor of value $(1/\alpha)$F. Similarly, $\tilde{F}_{2n}(s)/\tilde{F}_{2n-1}(s)$, $\tilde{L}_{2n+1}(s)/\tilde{L}_{2n}(s)$, $\tilde{L}_{2n}(s)/\tilde{L}_{2n-1}(s)$, $\tilde{L}_{2n}(s)/\tilde{F}_{2n}(s)$, and $\tilde{L}_{2n+1}(s)/\tilde{F}_{2n+1}(s)$ may all be realized by low-pass LC-ladder networks corresponding to Figs. 1(b), 1(c), 1(d), 1(e), and 1(f), respectively. Thus, we have the interesting result that $\tilde{F}_{n+1}(s)/\tilde{F}_n(s)$, $\tilde{L}_{n+1}(s)/\tilde{L}_n(s)$, and $\tilde{L}_n(s)/\tilde{F}_n(s)$ are all reactance functions. It is well known that the zeros and poles of a reactance function are simple, purely imaginary, and interlace [1]. Hence, the zeros of the polynomials $\tilde{F}_n(s)$ and $\tilde{L}_n(s)$ lie on the imaginary axis and are simple; further, the zeros of $\tilde{F}_n(s)$ and $\tilde{L}_n(s)$ interlace. Similar statements hold true for the zeros of $\tilde{F}_{n+1}(s)$ and $\tilde{F}_n(s)$, as well as those of $\tilde{L}_{n+1}(s)$ and $\tilde{L}_n(s)$.

Since, for the Pell and Pell-Lucas polynomials, we have

$$P_n(s) = F_n(2s) \tag{6a}$$

and

$$Q_n(s) = L_n(2s), \tag{6b}$$

it is obvious that $P_{n+1}(s)/P_n(s)$, $Q_{n+1}(s)/Q_n(s)$, and $Q_n(s)/P_n(s)$ are all reactance functions. In fact, using the frequency scaling theorem [1], it is seen that their realizations are the same as those of $F_{n+1}(s)/F_n(s)$, $L_{n+1}(s)/L_n(s)$, and $L_n(s)/F_n(s)$, respectively, except for a scaling of the values of the elements.

We now consider the case when $x=\beta$, a positive constant, and $y=s$; that is, we are dealing with $U_n(\beta, s)$ and $V_n(\beta, s)$. It is observed that when $\beta=1$ they reduce to the Jacobsthal polynomials [7]. Hence, we shall call $U_n(\beta, s)$ and $V_n(\beta, s)$ modified Jacobsthal polynomials and denote them by $\tilde{J}_n(s)$ and $\tilde{j}_n(s)$, respectively. It is then evident from the results of the previous section that $\tilde{J}_{2n+1}(s)/\tilde{J}_{2n}(s)$ may be realized as the DPI of the RL-network given by Fig. 1(a), where each of the series elements is a resistor of value β Ohms and each of the shunt elements is an inductor of value $(1/\beta)$H. Similarly, we can realize the functions $\tilde{J}_{2n}(s)/\tilde{J}_{2n-1}(s)$, $\tilde{j}_{2n+1}(s)/\tilde{j}_{2n}(s)$, $\tilde{j}_{2n}(s)/\tilde{j}_{2n-1}(s)$, $\tilde{j}_{2n}(s)/\tilde{J}_{2n}(s)$, and $\tilde{j}_{2n+1}(s)/\tilde{J}_{2n+1}(s)$ as DPIs of the RL-networks corresponding to Figs. 1(b), 1(c), 1(d), 1(e), and 1(f), respectively, where all the series elements are resistors and all the shunt elements are inductors. Thus, we have the result that $\tilde{J}_{n+1}(s)/\tilde{J}_n(s)$, $\tilde{j}_{n+1}(s)/\tilde{j}_n(s)$, and $\tilde{j}_n(s)/\tilde{J}_n(s)$ are all RL-impedance or RC-admittance functions. It is well known that the zeros and poles of an RL-impedance (or an RC-admittance) function lie on the negative real axis, are simple, and interlace; further, the one closest to the origin is a zero of the function [1]. Thus, the zeros of the polynomials $\tilde{J}_n(s)$ and $\tilde{j}_n(s)$ are real and negative; further, the zeros of $\tilde{J}_n(s)$ and $\tilde{j}_n(s)$ interlace, with the zero closest to the origin being that of $\tilde{j}_n(s)$. Similar statements hold true for the zeros of $\tilde{J}_{n+1}(s)$ and $\tilde{J}_n(s)$, as well as those of $\tilde{j}_{n+1}(s)$ and $\tilde{j}_n(s)$. It is also interesting to observe that $\tilde{J}_{n+1}(s)/\tilde{j}_n(s)$ is a ratio of two RC-admittance functions and hence, in general, is not realizable by two-element-kind networks; however, it is a positive real function (PRF), and so is always realizable by an RLC network. In fact, the zeros of $\tilde{J}_{n+1}(s)$ and $\tilde{j}_n(s)$ have a very interesting pairwise alternative relationship on the negative real axis [6].

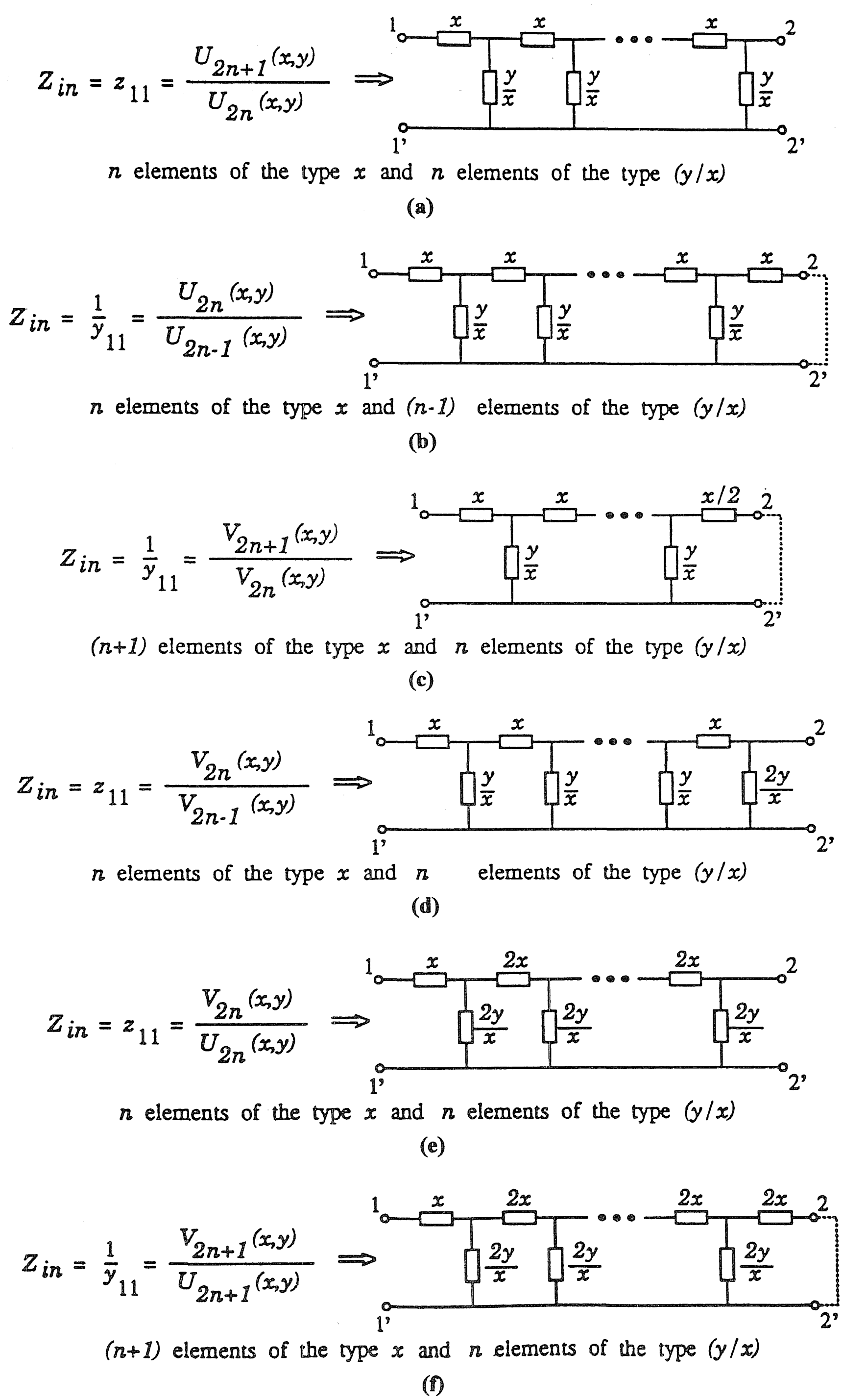

FIGURE 1. Various two-element-kind ladder networks.

4. LADDER TWO-PORTS

We will now express the chain parameters (see [2] and [14] for a definition of the chain parameters) of the six ladder two-port networks shown in Figs. 1(a)-1(f) in terms of the polynomials $U_n(x, y)$ and $V_n(x, y)$. First, consider the network of Fig. 1(a). We will now prove by induction that the chain matrix of this n-section ladder two-port is given by

$$[a_1]_n = \frac{1}{y^n}\begin{bmatrix} U_{2n+1}(x,y) & yU_{2n}(x,y) \\ U_{2n}(x,y) & yU_{2n-1}(x,y) \end{bmatrix}. \tag{7}$$

It is seen that, for $n = 1$, (7) holds since the chain matrix for one section [see Fig. 2(a)] is

$$[a_1]_1 = \begin{bmatrix} 1+(x^2/y) & x \\ (x/y) & 1 \end{bmatrix} = \frac{1}{y}\begin{bmatrix} U_3(x,y) & yU_2(x,y) \\ U_2(x,y) & yU_1(x,y) \end{bmatrix}. \tag{8}$$

The $(n+1)$-section ladder corresponding to Fig. 1(a) is shown in Fig. 2(b). Its chain matrix is

$$[a_1]_{n+1} = \frac{1}{y}\begin{bmatrix} x^2+y & xy \\ x & y \end{bmatrix}[a_1]_n. \tag{9}$$

Hence,

$$\begin{aligned}[a_1]_{n+1} &= \frac{1}{y^{n+1}}\begin{bmatrix} x(xU_{2n+1}+yU_{2n})+yU_{2n+1} & xy(xU_{2n}+yU_{2n-1})+y^2U_{2n} \\ xU_{2n+1}+yU_{2n} & xU_{2n}+yU_{2n-1} \end{bmatrix} \\ &= \frac{1}{y^{n+1}}\begin{bmatrix} xU_{2n+2}+yU_{2n+1} & y(xU_{2n+1}+yU_{2n}) \\ U_{2n+2} & yU_{2n+1} \end{bmatrix} = \frac{1}{y^{n+1}}\begin{bmatrix} U_{2n+3} & yU_{2n+2} \\ U_{2n+2} & yU_{2n+1} \end{bmatrix},\end{aligned}$$

where, for brevity, we have used U_n and V_n for $U_n(x, y)$ and $V_n(x, y)$. Hence, the result is true for $(n+1)$-sections; thus, the result given by (7) is established.

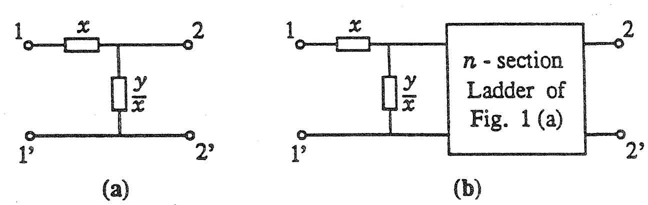

FIGURE 2. (a) One section of the ladder network of Fig. 1(a).
(b) An (n + 1)-section of the ladder of Fig. 1(a) considered as a cascade of the L-section of Fig. 2(a) and the n-section ladder of Fig. 1(a).

We will now obtain the chain matrix for the two-port of Fig. 1(b). This may be considered as a cascade of an $(n-1)$-section ladder of Fig. 1(a) and a single series element shown in Fig. 3. Hence, its chain matrix is given by

$$[a_2]_n = [a_1]_{n-1}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \frac{1}{y^{n-1}}\begin{bmatrix} U_{2n-1}(x,y) & yU_{2n-2}(x,y) \\ U_{2n-2}(x,y) & yU_{2n-3}(x,y) \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

Thus, the chain matrix of the two-port of Fig. 1(b) is given by

$$[a_2]_n = \frac{1}{y^{n-1}}\begin{bmatrix} U_{2n-1}(x,y) & U_{2n}(x,y) \\ U_{2n-2}(x,y) & U_{2n-1}(x,y) \end{bmatrix}. \tag{10}$$

Similarly, we can show that the chain matrix of the two-ports shown in Figs. 1(c) and 1(d) are, respectively, given by

$$[a_3]_n = \frac{1}{y^n}\begin{bmatrix} U_{2n+1}(x,y) & \frac{1}{2}V_{2n+1}(x,y) \\ U_{2n}(x,y) & \frac{1}{2}V_{2n}(x,y) \end{bmatrix} \tag{11}$$

and

$$[a_4]_n = \frac{1}{y^n}\begin{bmatrix} \frac{1}{2}V_{2n}(x,y) & yU_{2n}(x,y) \\ \frac{1}{2}V_{2n-1}(x,y) & yU_{2n-1}(x,y) \end{bmatrix}, \tag{12}$$

where relation (3) has been used.

The network of Fig. 1(e) can be considered as a cascade of an L-section and an $(n-1)$-section ladder of the type shown in Fig. 1(a), except that all the impedances are scaled by a factor of 2, as shown in Fig. 4. Hence, its chain matrix is given by

$$[a_5]_n = \frac{1}{y}\begin{bmatrix} (x^2/2)+y & xy \\ (x/2) & y \end{bmatrix} \frac{1}{y^{n-1}}\begin{bmatrix} U_{2n-1}(x,y) & 2yU_{2n-2}(x,y) \\ \frac{1}{2}U_{2n-2}(x,y) & yU_{2n-3}(x,y) \end{bmatrix}.$$

Thus, the chain matrix of the two-port of Fig. 1(e) may be expressed as

$$[a_5]_n = \frac{1}{y^n}\begin{bmatrix} \frac{1}{2}V_{2n}(x,y) & yV_{2n-1}(x,y) \\ \frac{1}{2}U_{2n}(x,y) & yU_{2n-1}(x,y) \end{bmatrix}. \tag{13}$$

Similarly, we can show that the chain matrix corresponding to the two-port of Fig. 1(f) is

$$[a_6]_n = \frac{1}{y^n}\begin{bmatrix} \frac{1}{2}V_{2n}(x,y) & V_{2n+1}(x,y) \\ \frac{1}{2}U_{2n}(x,y) & U_{2n+1}(x,y) \end{bmatrix}. \tag{14}$$

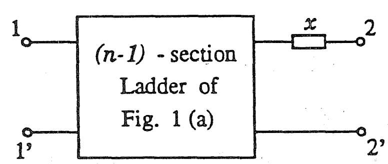

FIGURE 3. The ladder of Fig. 1(b) considered as a cascade of the *n*-section ladder of Fig. 1(a) and a series element.

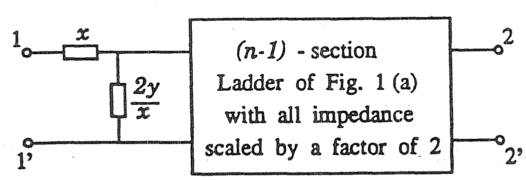

FIGURE 4. The ladder of Fig. 1(e) considered as a cascade of an *L*-section ladder of Fig. 1(a), which is suitably impedance-scaled.

As a consequence of the reciprocity property of these ladders, the determinants of the chain matrices given by (7), (10), (11), (12), (13), and (14) are all unity. Hence, we get the following interesting results:

$$U_{n+1}(x,y)U_{n-1}(x,y)-U_n^2(x,y)=(-1)^n y^{n-1}, \tag{15a}$$

$$U_{n+1}(x,y)V_n(x,y)-V_{n+1}(x,y)U_n(x,y)=(-1)^n 2y^n. \tag{15b}$$

As special cases, we also have

$$\widetilde{F}_{n+1}(s)\widetilde{F}_{n-1}(s)-\widetilde{F}_n^2(s)=(-1)^n\alpha^{n-1}, \tag{16a}$$

$$\widetilde{F}_{n+1}(s)\widetilde{L}_n(s)-\widetilde{L}_{n+1}(s)\widetilde{F}_n(s)=(-1)^n 2\alpha^n, \tag{16b}$$

and

$$\widetilde{J}_{n+1}(s)\widetilde{J}_{n-1}(s)-\widetilde{J}_n^2(s)=(-1)^n s^{n-1}, \tag{17a}$$

$$\widetilde{J}_{n+1}(s)\widetilde{j}_n(s)-\widetilde{j}_{n+1}(s)\widetilde{J}_n(s)=(-1)^n 2s^n. \tag{17b}$$

5. RELATIONS AMONG THE VARIOUS POLYNOMIALS

We first relate the two-variable polynomials $U_n(x,y)$ and $V_n(x,y)$ to the Morgan-Voyce polynomials $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$ (see [10], [14], [9], [8], [13]). It is known from [14] that the chain matrix of the network of Fig. 1(a) in terms of the Morgan-Voyce polynomials is given by

$$[a_1]_n=\begin{bmatrix} b_n(w) & xB_{n-1}(w) \\ \frac{x}{y}B_{n-1}(w) & b_{n-1}(w)\end{bmatrix}, \tag{18a}$$

where

$$w=x^2/y. \tag{18b}$$

Now, comparing (18a) and (5), we get

$$U_{2n+1}(x,y)=y^n b_n(x^2/y) \tag{19a}$$

and

$$U_{2n}(x,y)=xy^{n-1}B_{n-1}(x^2/y). \tag{19b}$$

Also, from (3), (19b), and (19a), we get

$$V_{2n+1}(x,y)=xy^n\{B_n(x^2/y)+B_{n-1}(x^2/y)\}$$

and

$$V_{2n}(x,y)=y^n\{b_n(x^2/y)+b_{n-1}(x^2/y)\}.$$

Hence,

$$V_{2n+1}(x,y)=xy^n c_n(x^2/y) \tag{19c}$$

and

$$V_{2n}(x,y)=y^n C_n(x^2/y). \tag{19d}$$

Using the above relations, (19a)-(19d), many interesting results for the two-variable polynomials $U_n(x,y)$ and $V_n(x,y)$—including the summation, product, and other formulas—may be derived from the properties of the Morgan-Voyce polynomials. However, we will not pursue it here. Instead, we establish the following relations among the various polynomials.

Case 1: Modified Fibonacci and Lucas Polynomials

Let $y = \alpha > 0$. Then, $U_n(x, \alpha) = \tilde{F}_n(x)$ and $V_n(x, \alpha) = \tilde{L}_n(x)$. Hence, from (19a)-(19d), we have

$$\tilde{F}_{2n+1}(x) = \alpha^n b_n(x^2 / \alpha), \quad \tilde{F}_{2n}(x) = \alpha^{n-1} x B_{n-1}(x^2 / \alpha), \tag{20a}$$

$$\tilde{L}_{2n+1}(x) = \alpha^n x c_n(x^2 / \alpha), \quad \tilde{L}_{2n}(x) = \alpha^n C_n(x^2 / \alpha). \tag{20b}$$

Of course, when $\alpha = 1$, the above reduce to the known relations between the Fibonacci, Lucas, and Morgan-Voyce polynomials.

Case 2: Modified Jacobsthal Polynomials

Let $\beta > 0$. Then $U_n(\beta, x) = \tilde{J}_n(x)$ and $V_n(\beta, x) = \tilde{j}_n(x)$. Hence, from (19a)-(19d), we have

$$\tilde{J}_{2n+1}(x) = x^n b_n(\beta^2 / x), \quad \tilde{J}_{2n}(x) = \beta x^{n-1} B_{n-1}(\beta^2 / x), \tag{21a}$$

$$\tilde{j}_{2n+1}(x) = \beta x^n c_n(\beta^2 / x), \quad \tilde{j}_{2n} = x^n C_n(\beta^2 / x). \tag{21b}$$

It is clear from (20) and (21) that the modified Fibonacci and Lucas polynomials and, hence, the Fibonacci and Lucas polynomials are directly related to the Jacobsthal polynomials by the simple relations

$$\tilde{F}_n(x) = x^{n-1} J_n(\alpha / x^2), \quad \tilde{L}_n(x) = x^n j_n(\alpha / x^2), \tag{22a}$$

and

$$F_n(x) = x^{n-1} J_n(1 / x^2), \quad L_n(x) = x^n j_n(1 / x^2). \tag{22b}$$

The above result could have been obtained from the networks of Figs. 1(e) and 1(f) which, respectively, realize $j_{2n}(s) / J_{2n}(s)$ and $j_{2n+1}(s) / J_{2n+1}(s)$ when $x = 1$ and $y = s$, by first transforming the complex frequency from s to α / s^2, and then multiplying all the resulting impedances by s.

Case 3: Modified Chebyshev Polynomials

We define $\tilde{G}_n(x)$ and $\tilde{H}_n(x)$, the modified Chebyshev polynomials of the first and second kind, respectively, by

$$\tilde{G}_n(x) = U_n(x, -\alpha), \quad \tilde{H}_n(x) = V_n(x, -\alpha), \tag{23a}$$

where

$$\alpha > 0. \tag{23b}$$

Then, from (19a)-(19d), we have

$$\tilde{G}_{2n+1}(x) = (-1)^n \alpha^n b_n(-x^2 / \alpha), \quad \tilde{G}_n(x) = (-1)^{n-1} \alpha^{n-1} x B_{n-1}(x^2 / \alpha) \tag{24a}$$

and

$$\tilde{H}_{2n+1}(x) = (-1)^n \alpha^n x c_n(-x^2 / \alpha), \quad \tilde{H}_{2n}(x) = (-1)^n \alpha^n C_n(-x^2 / \alpha). \tag{24b}$$

Now, using (21a) and (21b), we may relate the modified Chebyshev polynomials directly to the Jacobsthal polynomials by

$$\tilde{G}_n(x) = x^{n-1} J_n(-\alpha / x^2), \quad \tilde{H}_n(x) = x^n j_n(-\alpha / x^2). \tag{25}$$

Now $\Phi_n(x)$ and $\Theta_n(x)$, the Fermat polynomials of the first and second kinds, respectively, are obtained by letting $\alpha = 2$ in (23). Hence,

$$\Phi_n(x) = x^{n-1}J_n(-2/x^2), \quad \Theta_n(x) = x^n j_n(-2/x^2). \tag{26}$$

Also, the Chebyshev polynomials $S_n(x)$ and $T_n(x)$ are given by

$$S_n(x) = U_n(2x, -1) = (2x)^{n-1}J_n(-1/4x^2) \tag{27a}$$

and

$$T_n(x) = \tfrac{1}{2}V_n(2x, -1) = 2^{n-1}x^n j_n(-1/4x^2). \qquad 27b)$$

Case 4: Brahmagupta's Polynomials

Brahmagupta's polynomials $x_n(x, y)$ and $y_n(x, y)$ are defined as follows (see [12]):

$$x_{n+1}(x, y) = 2xx_n(x, y) - \lambda x_{n-1}(x, y), \quad x_0 = 1, \ x_1 = x, \tag{28a}$$

and

$$y_{n+1}(x, y) = 2xy_n(x, y) - \lambda y_{n-1}(x, y), \quad y_0 = 1, \ y_1 = y. \tag{28b}$$

It is known that if (x_1, y_1) is a positive integer set satisfying the relation

$$x_1^2 - ty_1^2 = \lambda \tag{29a}$$

where t is a square-free integer, then the positive integer set (x_n, y_n) is a solution of Brahmagupta-Bhaskara's equation given by [15]:

$$x_n^2 - ty_n^2 = \lambda^n. \tag{29b}$$

The Brahmagupta polynomials are related to $U_n(x, y)$ and $V_n(x, y)$ by

$$x_n(x, y) = \tfrac{1}{2}V_n(2x, -\lambda), \quad y_n(x, y) = yU_n(2x, -\lambda), \tag{30a}$$

and to the Jacobsthal polynomials by

$$x_n(x, y) = 2^{n-1}x^n j_n(-\lambda/4x^2), \quad y_n(x, y) = y(2x)^{n-1}J_n(-\lambda/4x^2). \tag{30b}$$

If $\lambda > 0$, and say $= \alpha$, then

$$x_n(x, y) = \tfrac{1}{2}\widetilde{H}_n(2x), \quad y_n(x, y) = y\widetilde{G}_n(2x). \tag{31}$$

However, if $\lambda < 0$, say $= -\alpha$, $\alpha > 0$, then

$$x_n(x, y) = \tfrac{1}{2}\widetilde{L}_n(2x), \quad y_n(x, y) = y\widetilde{F}_n(2x). \tag{32}$$

Of course, the polynomials $\widetilde{G}_n(x)$, $\widetilde{H}_n(x)$, $\widetilde{F}_n(x)$, and $\widetilde{L}_n(x)$ are related to the Jacobsthal and Morgan-Voyce polynomials, and hence we may relate the Brahmagupta polynomials also to these polynomials. Finally, it is seen that, when $\lambda = 1$,

$$x_n(x, y) = T_n(x), \quad y_n(x, y) = yS_n(x), \tag{33}$$

while, when $\lambda = -1$,

$$x_n(x, y) = \tfrac{1}{2}Q_n(x), \quad y_n(x, y) = yP_n(x). \tag{34}$$

As a consequence of (33) and (34), we can show that

$$Q_{2n}(x) = 2T_n(2x^2 + 1), \quad P_{2n}(x) = 2xS_n(2x^2 + 1). \tag{35}$$

6. DERIVATIVE POLYNOMIALS AND THEIR ZEROS

In this section we will show that we can get information about the location of the zeros of the derivative polynomials using the following known results about the nature of the impedance functions of two-element-kind networks.

Property 1: If the driving point impedance $Z(s) = N(s) / D(s)$ is a reactance function, then so is $Z_1(s) = N'(s) / D'(s)$, where the prime indicates the derivative with respect to s.

Property 2: If $Z(s) = N(s) / D(s)$ is an RL-impedance (or an RC-admittance) function, then so is $Z_1(s) = N'(s) / D'(s)$.

Let us first consider the function, $Z(s) = \widetilde{L}_n(s) / \widetilde{F}_n(s)$, a ratio of the modified Fibonacci and Lucas polynomials. We have shown in Section 3 that $Z(s)$ is a reactance function. Hence, from Property 1, the function $Z_1(s) = \widetilde{L}_n'(s) / \widetilde{F}_n'(s)$ is also a reactance function. By successively applying Property 1 k times, we see that the function $Z_k(s) = \widetilde{L}_n^{(k)}(s) / \widetilde{F}_n^{(k)}(s)$, where (k) represents the k^{th} derivative with respect to s, is also a reactance function. Using the property of reactance functions, we see that the zeros of $\widetilde{L}_n^{(k)}(s)$ and $\widetilde{F}_n^{(k)}(s)$ are simple and lie on the imaginary axis, with the two sets of zeros interlacing with each other. Similar statements hold for the zeros of $\widetilde{L}_{n+1}^{(k)}(s)$ and $\widetilde{L}_n^{(k)}(s)$, as well as for those of $\widetilde{F}_{n+1}^{(k)}(s)$ and $\widetilde{F}_n^{(k)}(s)$.

We also proved in Section 3 that the ratios $\widetilde{j}_n(s) / \widetilde{J}_n(s)$, $\widetilde{J}_{n+1}(s) / \widetilde{J}_n(s)$, and $\widetilde{j}_{n+1}(s) / \widetilde{j}_n(s)$ are all RL-impedance functions. Thus, from Property 2, we see that $\widetilde{j}_n^{(k)}(s) / \widetilde{J}_n^{(k)}(s)$, $\widetilde{j}_{n+1}^{(k)}(s) / \widetilde{j}_n^{(k)}(s)$, and $\widetilde{J}_{n+1}^{(k)}(s) / \widetilde{J}_n^{(k)}(s)$ are also RL-impedance functions. Using the property of RL-impedance functions, we see that the zeros of $\widetilde{J}_n^{(k)}(s)$ and $\widetilde{j}_n^{(k)}(s)$ are real and negative. Further, the zeros of $\widetilde{J}_n^{(k)}(s)$ interlace with those of $\widetilde{j}_n^{(k)}(s)$, with the zeros closest to the origin being that of $\widetilde{j}_n^{(k)}(s)$. Similar statements hold true for the zeros of $\widetilde{J}_{n+1}^{(k)}(s)$ and $\widetilde{J}_n^{(k)}(s)$, as well as those of $\widetilde{j}_{n+1}^{(k)}(s)$ and $\widetilde{j}_n^{(k)}(s)$.

Similar results may be established regarding the zeros of the derivatives of the Morgan-Voyce polynomials.

7. CONCLUDING REMARKS

It is shown that there exists a close relationship between the network functions of LC, RL, and RC ladder networks and certain generalized polynomials. In view of this, many interesting properties of these polynomials may be derived using the well-known properties of two-element-kind ladder networks, and vice-versa. A few elegant results regarding the location of the zeros of the polynomials such as the Fibonacci, Lucas, Jacobsthal, as well as their derivative polynomials have been derived. Also, the interrelations among these various polynomials and the Morgan-Voyce polynomials have been derived.

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