PHASED TILINGS AND GENERALIZED FIBONACCI IDENTITIES

Arthur T. Benjamin
Dept. of Mathematics, Harvey Mudd College, Claremont, CA 91711
benjamin@hmc.edu

Jennifer J. Quinn
Dept. of Mathematics, Occidental College, Los Angeles, CA 90041
jquinn@oxy.edu

Francis Edward Su
Dept. of Mathematics, Harvey Mudd College, Claremont, CA 91711
su@math.hmc.edu

(*Submitted August 1998-Final Revision May 1999*)

1. INTRODUCTION

Fibonacci numbers arise in the solution of many combinatorial problems. They count the number of binary sequences with no consecutive zeros, the number of sequences of 1's and 2's which sum to a given number, and the number of independent sets of a path graph. Similar interpretations exist for Lucas numbers. Using these interpretations, it is possible to provide combinatorial proofs that shed light on many interesting Fibonacci and Lucas identities (see [1], [3]). In this paper we extend the combinatorial approach to understand relationships among generalized Fibonacci numbers.

Given G_0 and G_1, *a generalized Fibonacci sequence* $G_0, G_1, G_2, \ldots$ is defined recursively by $G_n = G_{n-1} + G_{n-2}$ for $n \geq 2$. Two important special cases are the classical Fibonacci sequence F_n ($F_0 = 0$ and $F_1 = 1$) and the Lucas sequence L_n ($L_0 = 2$ and $L_1 = 1$).

These sequences satisfy numerous relationships. Many are documented in Vajda [6], where they are proved by algebraic means. Our goal is to *recount* these identities by combinatorial means. We introduce several combinatorial techniques which allow us to provide new proofs of nearly all the identities in [6] involving generalized Fibonacci numbers. We show that in the framework of *phased tilings*, these identities follow naturally as the tilings are counted, represented, and transformed in clever ways. These techniques are developed in the next several sections. In the final section, we discuss possible extensions.

2. COMBINATORIAL INTERPRETATION

Recall that F_{n+1} counts the number of sequences of 1's and 2's which sum to n. Equivalently, F_{n+1} counts the number of ways to tile a $1 \times n$ rectangle (called an *n-board* consisting of *cells* labeled $1, \ldots, n$) with 1×1 *squares* and 1×2 *dominoes*. For combinatorial convenience, we define $f_n = F_{n+1}$. Then f_n is the number of ways to tile an n-board with squares and dominoes.

When G_0 and G_1 are nonnegative integers, we shall obtain an analogous combinatorial interpretation of the generalized Fibonacci numbers G_n. Define a *phased n-tiling* to be a tiling of an n-board by squares and dominoes in which the last tile is distinguished in a certain way. Specifically, if the last tile is a domino, it can be assigned one of G_0 possible *phases*, and if the last tile is a square, it can be assigned one of G_1 possible phases. For example, when $G_0 = 5$ and $G_1 = 17$, there are $G_3 = 39$ phased tilings of length 3 as follows: There are 5 of the form (square, phased domino); 17 of the form (domino, phased square); and 17 of the form (square, square, phased square). In general, let $g_0 = G_0$, $g_1 = G_1$, and, for $n \geq 2$, let g_n count the number of phased n-tilings. By conditioning on whether the first tile is a square or domino, we obtain the identity $g_n = g_{n-1} + g_{n-2}$ for $n \geq 2$. Hence, $g_n = G_n$, giving the desired interpretation.

This combinatorial definition can be extended to $n = 1$ and $n = 0$. Clearly G_1 counts the number of phased 1-tilings. It will be convenient to assign the "last" tile of a 0-board one of the G_0 domino phases.

Notice when there exists only one domino phase and only one square phase, we recover our original interpretation of f_n.

Previous interpretations of the Lucas numbers L_n (see [1], [4], [5]) counted the number of ways to tile a "circular" n-board by squares or dominoes. Since $L_0 = 2$ and $L_1 = 1$, a phased n-board tiling can end with a phase one domino, a phase two domino, or a phase one square. In all three cases, the corresponding circular n-board tiling arises by first gluing cells n and 1 together. Tilings that end in a phase two domino are then rotated one cell to obtain a circular tiling with a domino covering cells n and 1.

3. ELEMENTARY IDENTITIES

Before launching into more sophisticated techniques, we demonstrate how our combinatorial interpretation of G_n yields quick proofs of some basic identities. For instance, by conditioning on whether the last tile is a phased domino or a phased square, we immediately obtain, for $n \geq 2$, $G_n = G_0 f_{n-2} + G_1 f_{n-1}$.

More identities are obtained by conditioning on other events. Consider

Identity 1 [Vajda (33)]: $\sum_{k=0}^{n} G_k = G_{n+2} - G_1$.

The right-hand side of this equality counts all phased $(n+2)$-tilings containing at least one domino (there are G_1 phased tilings consisting of all squares). The left-hand side is obtained by conditioning on the position of the first domino. If the first domino covers cells $n-k+1$ and $n-k+2$ $(0 \leq k \leq n)$, then the preceding cells are covered by squares and the remaining cells can be covered G_k ways.

Similarly, there are $G_{2n} - G_0$ phased $2n$-tilings with at least one square. By conditioning on the position of the first square, we obtain

Identity 2 [Vajda (34)]: $\sum_{k=1}^{n} G_{2k-1} = G_{2n} - G_0$.

A phased $(2n+1)$-tiling must contain a first square, which leads to

Identity 3 [Vajda (35)]: $G_1 + \sum_{k=1}^{n} G_{2k} = G_{2n+1}$.

The G_1 term on the left-hand side counts those boards that begin with n dominoes followed by a phased square.

To prove

Identity 4 [Vajda (8)]: $G_{m+n} = f_m G_n + f_{m-1} G_{n-1}$, we consider whether or not a phased $(m+n)$-tiling can be separated into an (unphased) m-tiling followed by a phased n-tiling. There are $f_m G_n$ tilings breakable at cell m. The unbreakable tilings must contain a domino covering cells m and $m+1$; the remaining board can be covered $f_{m-1}G_{n-1}$ ways.

4. BINOMIAL IDENTITIES

Vajda contains several identities involving generalized Fibonacci numbers and binomial coefficients. All of these are special cases of the following two identities.

Identity 5 [Vajda (46)]: $G_{n+p} = \sum_{i=0}^{p} \binom{p}{i} G_{n-i}$.

Identity 6 [Vajda (66)]: $G_{m+(t+1)p} = \sum_{i=0}^{p} \binom{p}{i} f_t^i f_{t-1}^{p-i} G_{m+i}$.

When $n \geq p$, Identity 5 counts phased $(n+p)$-tilings by conditioning on the number of dominoes that appear among the first p tiles. Given an initial segment of i dominoes and $p-i$ squares, $\binom{p}{i}$ counts the number of ways to select the i positions for the dominoes among the first p tiles. G_{n-i} counts the number of ways the remaining $n-i$ cells can be given a phased tiling.

Identity 6 can be seen as trying to break a phased $(m+(t+1)p)$-tiling into p unphased segments of length t followed by a phased remainder. The first segment consists of the tiles covering cells 1 through j_1, where $j_1 = t$ if the tiling is breakable at cell t and $j_1 = t+1$ otherwise. The next segment consists of the tiles covering cells j_1+1 through j_1+j_2, where $j_2 = t$ if the tiling is breakable at cell j_1+t and $j_2 = t+1$ otherwise. Continuing in this fashion, we decompose our phased tiling into p tiled segments of length t or $t+1$ followed by a phased remainder of length at least m. Since the length $t+1$ segments must end with a domino, the term $\binom{p}{i} f_t^i f_{t-1}^{p-i} G_{m+i}$ counts the number of phased $(m+(t+1)p)$-tilings with exactly i segments of length t.

5. SIMULTANEOUS TILINGS

Identities involving squares of generalized Fibonacci numbers suggest investigating pairs of phased tilings. The right-hand side of

Identity 7 [Vajda (39)]: $\sum_{i=1}^{2n} G_{i-1}G_i = G_{2n}^2 - G_0^2$

counts ordered pairs (A, B) of phased $2n$-tilings, where A or B contains at least one square. To interpret the left-hand side, we define the parameter k_X to be the first cell of the phased tiling X covered by a square. If X is all dominoes, we set k_X equal to infinity. Since, in this case, at least one square exists in (A, B), the minimum of k_A and k_B must be finite and odd. Let $k = \min\{k_A, k_B+1\}$. When k is odd, A and B have dominoes covering cells 1 through $k-1$ and A has a square covering cell k. Hence, the number of phased pairs (A, B) with odd k is $G_{2n-k}G_{2n-k+1}$. When k is even, A has dominoes covering cells 1 through k and B has dominoes covering cells 1 through $k-2$ with a square covering cell $k-1$. Hence, the number of phased pairs (A, B) with even k is also $G_{2n-k}G_{2n-k+1}$. Setting $i = 2n+1-k$ gives the desired identity.

Similarly, the next identity counts ordered pairs of phased $(2n+1)$-tilings that contain an unphased square. Conditioning on the first unphased square yields

Identity 8 [Vajda (41)]: $\sum_{i=2}^{2n+1} G_{i-1}G_i = G_{2n+1}^2 - G_1^2$.

In the same spirit, our next identity conditions on the location of the first domino in a pair of phased tilings.

Identity 9 [Vajda (43)]: $\sum_{i=1}^{n} G_{i-1}G_{i+2} = G_{n+1}^2 - G_1^2$.

The right-hand side counts the number of pairs (A, B) of phased $(n+1)$-tilings, where A or B contains at least one domino. Here we define the parameter ℓ_X to be the first cell of the phased tiling X covered by a domino. If X is all squares, we set ℓ_X equal to infinity. Let $\ell = \min\{\ell_A, \ell_B\}$. The number of phased $(n+1)$-tiling pairs (A, B), where the ℓ^{th} cell of A is covered by a domino is $G_{n-\ell}G_{n-\ell+2}$ and the number of such pairs where the ℓ^{th} cell of A is covered by a square is $G_{n-\ell+1}G_{n-\ell}$. This implies

$$\sum_{\ell=1}^{n} G_{n-\ell}(G_{n-\ell+2} + G_{n-\ell+1}) = G_{n+1}^2 - G_1^2$$

Substituting $G_{n-\ell+3}$ for $G_{n-\ell+2} + G_{n-\ell+1}$ and letting $i = n - \ell + 1$ yields the desired identity.

6. A TRANSFER PROCEDURE

The identities proved in this section all take advantage of the same technique. Before proceeding, we introduce helpful notation. For $m \geq 0$, define $\mathcal{G}_m$ to be the set of all phased m-tilings with G_0 domino phases and G_1 square phases. An element $A \in \mathcal{G}_m$ created from a sequence of e_1 dominoes, e_2 squares, e_3 dominoes, ..., and ending with a phased tile can be expressed uniquely as $A = d^{e_1}s^{e_2}d^{e_3}s^{e_4}\cdots d^{e_{t-1}}s^{e_t}p$, where p represents the phase of the last tile. All exponents are positive except that e_1 or e_t may be 0, and $2e_1 + e_2 + 2e_3 + e_4 + \cdots + 2e_{t-1} + e_t = m$. When $e_t = 0$, the last tile is a domino and $p \in \{1, \ldots, G_0\}$; when $e_t \geq 1$, the last tile is a square and $p \in \{1, \ldots, G_1\}$. Likewise, for $n \geq 0$, define $\mathcal{H}_n$ to be the set of all phased n-tilings with H_0 domino phases and H_1 square phases. Notice that the sizes of $\mathcal{G}_m$ and $\mathcal{H}_n$ are G_m and H_n, respectively.

We introduce a transfer procedure T to map an ordered pair $(A, B) \in \mathcal{G}_m \times \mathcal{H}_n$ to an ordered pair $(A', B') \in \mathcal{G}_{m-1} \times \mathcal{H}_{n+1}$, where $1 \leq m \leq n$. T has the effect of shrinking the smaller tiling and growing the larger tiling by one unit. For such a pair (A, B), define $k = \min\{k_A, k_B\}$, the first cell in A or B that is covered by a square. If the k^{th} cell of A is covered by a square and $1 \leq k \leq m-1$, then we transfer that square from A to the k^{th} cell of B. Formally, before the transfer, we have $A = d^{(k-1)/2}sa$, $B = d^{(k-1)/2}b$, where $a \in \mathcal{G}_{m-k}$, $b \in \mathcal{H}_{n-k+1}$. The transfer yields $A' = d^{(k-1)/2}a$, $B' = d^{(k-1)/2}sb$. If the k^{th} cell of A is covered by a domino and $1 \leq k \leq m-2$, then we exchange that domino with the square in the k^{th} cell of B. Formally, before the exchange, $A = d^{(k+1)/2}a$, $B = d^{(k-1)/2}sb$, where $a \in \mathcal{G}_{m-k-1}$, $b \in \mathcal{H}_{n-k}$. The exchange yields $A' = d^{(k-1)/2}sa$, $B' = d^{(k+1)/2}b$. We abbreviate this transformation by $T(A, B) = (A', B')$. Notice that our rules do not allow for a phased tile to be transferred or exchanged.

Lemma 1: For $1 \leq m \leq n$, T establishes an *almost* one-to-one correspondence between $\mathcal{G}_m \times \mathcal{H}_n$ and $\mathcal{G}_{m-1} \times \mathcal{H}_{n+1}$. The difference of their sizes satisfies

$$G_mH_n - G_{m-1}H_{n+1} = (-1)^m[G_0H_{n-m+2} - G_1H_{n-m+1}].$$

Proof: Notice that when T is defined, $T(A, B)$ has the same k value as (A, B), which makes T easy to reverse. It remains to enumerate $(A, B) \in \mathcal{G}_m \times \mathcal{H}_n$ for which T is undefined and $(A', B') \in \mathcal{G}_{m-1} \times \mathcal{H}_{n+1}$ that do not appear in the image of T.

When m is odd, T is undefined whenever $k = m$. Here $A = d^{(m-1)/2}a$, $B = d^{(m-1)/2}b$, where $a \in \mathcal{G}_1$, $b \in \mathcal{H}_{n-m+1}$. Hence, the domain of T contains $G_m H_n - G_1 H_{n-m+1}$ elements. The elements of $\mathcal{G}_{m-1} \times \mathcal{H}_{n+1}$ that do not appear in the image of T have $k \geq m$ and are therefore of the form $A' = d^{(m-1)/2}p$, $B' = d^{(m-1)/2}b'$, where $p \in \{1, \dots, G_0\}$, $b' \in \mathcal{H}_{n-m+2}$. Hence, the image of T consists of $G_{m-1}H_{n+1} - G_0 H_{n-m+2}$ elements. Since T is one-to-one we have, when m is odd,

$$G_m H_n - G_1 H_{n-m+1} = G_{m-1}H_{n+1} - G_0 H_{n-m+2}.$$

When m is even, T is undefined whenever $k \geq m$ and sometimes undefined when $k = m-1$. Specifically, T is undefined when $A = d^{m/2}p$, $B = d^{(m-2)/2}b$, where $p \in \{1, \dots, G_0\}$, $b \in \mathcal{H}_{n-m+2}$. Hence, the domain of T contains $G_m H_n - G_0 H_{n-m+2}$ elements. The elements of $\mathcal{G}_{m-1} \times \mathcal{H}_{n+1}$ that do not appear in the image are of the form $A' = d^{(m-2)/2}a'$, $B' = d^{m/2}b'$, where $a' \in \mathcal{G}_1$, $b' \in \mathcal{H}_{n-m+1}$. Hence, the image of T consists of $G_{m-1}H_{n+1} - G_1 H_{n-m+1}$ elements. Thus, when m is even, we have

$$G_m H_n - G_0 H_{n-m+2} = G_{m-1}H_{n+1} - G_1 H_{n-m+1}. \quad \square$$

We specialize Lemma 1 by setting $m = n$ and choosing the same initial conditions for $\mathcal{G}_n$ and $\mathcal{H}_n$ to obtain

Identity 10 [Vajda (28)]: $G_{n+1}G_{n-1} - G_n^2 = (-1)^n(G_1^2 - G_0 G_2)$.

Alternately, setting $G_m = F_m$ and evaluating lemma 1 at $m+1$, we obtain

Identity 11 [Vajda (9)]: $H_{n-m} = (-1)^m(F_{m+1}H_n - F_m H_{n+1})$ for $0 \leq m \leq n$.

A slightly different transfer process is used to prove

Identity 12 [Vajda (10a)]: $G_{n+m} + (-1)^m G_{n-m} = L_m G_n$ for $0 \leq m \leq n$.

We construct an almost one-to-one correspondence from $\mathcal{L}_m \times \mathcal{G}_n$ to $\mathcal{G}_{m+n}$, where $\mathcal{L}_m$ denotes the set of Lucas tilings of length m. Let $(A, B) \in \mathcal{L}_m \times \mathcal{G}_n$. If A ends in a (phase 1) square or a phase 1 domino, then we simply append A to the front of B to create an $(m+n)$-tiling that is breakable at m. Otherwise, A ends in a phase 2 domino. In this case, before appending A to the front of B, we transfer a unit from B to A by a similar process. (If the first square occurs in B, then transfer it into the corresponding cell of A. Otherwise, the first square of A is exchanged with the corresponding domino in B.) This creates a tiling of $\mathcal{G}_{m+n}$ that is unbreakable at m. When m is even, the transfer is undefined for the G_{n-m} elements of $\mathcal{L}_m \times \mathcal{G}_n$, where A contains only dominoes, ending with a phase 2 domino, and B begins with $m/2$ dominoes. Otherwise, the transfer is one-to-one and onto $\mathcal{G}_{m+n}$. When m is odd, the transfer is always defined but misses the G_{n-m} elements of $\mathcal{G}_{m+n}$ that begin with m dominoes. Identity 12 follows.

A similar argument establishes

Identity 13 [Vajda (10b)]: $G_{n+m} - (-1)^m G_{n-m} = F_m(G_{n-1} + G_{n+1})$ for $0 \leq m \leq n$.

The transfer process T can be refined to allow us to shrink and grow pairs of phased tilings by more than one unit. Specifically, we construct an almost one-to-one correspondence between $\mathcal{G}_m \times \mathcal{H}_n$ and $\mathcal{G}_{m-h} \times \mathcal{H}_{n+h}$, where $1 \leq h \leq m-1 < n$. Let $\mathcal{F}_n$ denote the set of (unphased) n-tilings.

So $|\mathcal{F}_n| = f_n$. Given $(A, B) \in \mathcal{G}_m \times \mathcal{H}_n$, define a transfer process T_h as follows: if A is breakable at cell h, i.e., $A = a_1 a_2$, where $a_1 \in \mathcal{F}_h$ and $a_2 \in \mathcal{G}_{m-h}$, then we append segment a_1 to the beginning of B. That is to say, $T_h(A, B) = (A', B')$, where $A' = a_2$ and $B' = a_1 B$. If A is unbreakable at cell h, i.e., $A = a_1 d a_2$, where $a_1 \in \mathcal{F}_{h-1}$ and $a_2 = \mathcal{G}_{m-h-1}$, then let $(A'', B'') = T(da_2, B)$ and $(A', B') = (A'', a_1 B'')$. Notice that B'', when defined, will necessarily begin with a domino and, therefore, B' will be unbreakable at h.

Discrepancies in T_h mapping $\mathcal{G}_m \times \mathcal{H}_n$ to $\mathcal{G}_{m-h} \times \mathcal{H}_{n+h}$, are proportional (by a factor of f_{h-1}) to the discrepancies in T mapping $\mathcal{G}_{m-h+1} \times \mathcal{H}_n$ to $\mathcal{G}_{m-h} \times \mathcal{H}_{n+1}$. Hence, for $1 \le h \le m-1$, Lemma 1 implies

$$G_m H_n - G_{m-h} H_{n+h} = (-1)^{m-h+1} f_{h-1}(G_0 H_{n-m+h+1} - G_1 H_{n-m+h}). \tag{1}$$

Notice that $f_{h-1} H_{n-m+h+1}$ counts the number of phased $(n-m+2h)$-tilings that are breakable at $h-1$. Hence, $f_{h-1} H_{n-m+h+1} = H_{n-m+2h} - f_{h-2} H_{n-m+h}$. Similarly, $f_{h-1} G_1 = G_h - f_{h-2} G_0$. So equation (1) can be rewritten as

$$G_m H_n - G_{m-h} H_{n+h} = (-1)^{m-h+1}(G_0 H_{n-m+2h} - G_h H_{n-m+h}).$$

Reindexing, this is equivalent to

Identity 14 [Vajda (18)]: $G_{n+h} H_{n+k} - G_n H_{n+h+k} = (-1)^n (G_h H_k - G_0 H_{h+k})$.

This identity is applied, directly or indirectly, by Vajda to obtain identities (19a) through (32).

7. BINARY SEQUENCES

There are identities involving generalized Fibonacci numbers and powers of 2. This leads us to investigate the relationship between binary sequences and Fibonacci tilings.

A binary sequence $x = x_1 x_2 \ldots x_n$ can be viewed as a set of instructions for creating a Fibonacci tiling of length less than or equal to n. Reading x from left to right, we interpret 1's and 01's as squares and dominoes, respectively. The construction halts on encountering a 00 or the end of the sequence. For example, 111010110101 represents the 12-tiling $s^3 d^2 s d^2$, 1110101110 represents the 9-tiling $s^3 d^2 s^2$, and 0111001011 represents the 4-tiling ds^2. Binary sequences that begin with 00 denote the 0-tiling.

Given n, tilings of length n and $n-1$ are represented uniquely by binary sequences of length n that end with a 1 or 0, respectively. For $k \le n-2$, a k-tiling is represented by 2^{n-k-2} binary sequences of length n since the first $k+2$ bits are determined by the k-tiling followed by 00; the remaining $n-(k+2)$ bits may be chosen freely. This yields the following identity:

$$f_n + f_{n-1} + \sum_{k=0}^{n-2} f_k 2^{n-k-2} = 2^n. \tag{2}$$

By dividing by 2^n, reindexing, and employing $f_{n+1} = f_n + f_{n-1}$, we obtain

Identity 15 [Vajda (37a)]: $\sum_{k=2}^{n} \frac{f_{k-2}}{2^k} = 1 - \frac{f_{n+1}}{2^n}$.

The same strategy can be applied to phased tilings. Here, for convenience, we assume the phase is determined by the first tile (rather than the last). The phased identity corresponding to equation (2) is

$$G_{n+1} + G_n + \sum_{k=0}^{n-1} G_k 2^{n-k-1} = 2^n (G_0 + G_1). \tag{3}$$

The right-hand side counts the number of ways to select a length n binary sequence x and a phase p. From this, we construct a length $n+1$ binary sequence. If p is a domino phase, construct the sequence $0x$; if p is a square phase, construct the sequence $1x$. Interpret this new $n+1$-sequence as a Fibonacci tiling in the manner discussed previously, and assign the tiling the phase p. By construction, the phase is compatible with the first tile. (Recall that empty tilings are assigned a domino phase.) A phased tiling of length $n+1$ or n has a unique (x, p) representation. For $0 \le k \le n-1$, a phased k-tiling has 2^{n-k-1} representations. This establishes equation (3). Dividing by 2^n gives

Identity 16 [Vajda (37)]: $\sum_{k=0}^{n-1} \frac{G_k}{2^{k+1}} = (G_0 + G_1) - \frac{G_{n+1} + G_n}{2^n}.$

8. DISCUSSION

The techniques presented in this paper are simple but powerful—counting phased tilings enables us to give visual interpretations to expressions involving generalized Fibonacci numbers. This approach facilitates a clearer understanding of existing identities, and can be extended in a number of ways.

For instance, by allowing tiles of length 3 or longer, we can give combinatorial interpretation to higher-order recurrences; however, the initial conditions do not work out so neatly, since the number of phases that the last tile admits do not correspond with the initial conditions of the recurrence.

Another possibility is to allow *every* square and domino to possess a number of phases, depending on its location. This leads to recurrences of the form $x_n = a_n x_{n-1} + b_n x_{n-2}$. The special case where $b_n = 1$ for all n provides a tiling interpretation of the numerators and denominators of simple finite continued fractions and is treated in [2].

ACKNOWLEDGMENT

The authors gratefully acknowledge the anonymous referee for many valuable suggestions.

REFERENCES

1. A. T. Benjamin & J. J. Quinn. "Recounting Fibonacci and Lucas Identities." *College Math. J.* **30.5** (1999):359-66.
2. A. T. Benjamin, J. J. Quinn, & F. E. Su. "Counting on Continued Fractions." *Math. Magazine* **73.2** (2000):98-104.
3. R. C. Brigham, R. M. Caron, P. Z. Chinn, & R. P. Grimaldi. "A Tiling Scheme for the Fibonacci Numbers." *J. Recreational Math.* **28.1** (1996-1997):10-16.
4. L. Comtet. *Advanced Combinatorics: The Art of Finite and Infinite Expansions*. Dordrecht: D. Reidel, 1974.
5. H. Prodinger & R. F. Tichy. "Fibonacci Numbers of Graphs." *The Fibonacci Quarterly* **20.1** (1982):16-21.
6. S. Vajda. *Fibonacci and Lucas Numbers, and the Golden Section*. New York: Wiley, 1989.

AMS Classification Numbers: 05A19, 11B39

❖❖❖