GENERALIZED JACOBSTHAL POLYNOMIALS

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1. INTRODUCTION

In this paper we study two classes of polynomials: the generalized Jacobsthal polynomials $\{J_{n,m}(x)\}$ and the generalized Jacobsthal-Lucas polynomials $\{j_{n,m}(x)\}$ defined, respectively, by

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x), \quad n \geq m, \tag{1.1}$$

with $J_{0,m}(x) = 0$, $J_{n,m}(x) = 1$, $n = 1, 2, \ldots, m-1$, and

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x), \quad n \geq m, \tag{1.2}$$

with $j_{0,m}(x) = 2$, $j_{n,m}(x) = 1$, $n = 1, 2, \ldots, m-1$. In this paper we call these polynomials the generalized Jacobsthal polynomials.

The polynomials $J_{n,2}(x)$ and $j_{n,2}(x)$ are studied in [4].

For $m = 2$ and $x = 1$, we get the Jacobsthal numbers $\{J_{n,2}(1)\}$ and Jacobsthal-Lucas numbers $\{j_{n,2}(1)\}$, which are studied in [3].

Here we shall prove the list of characteristic properties of the polynomials $\{J_{n,m}(x)\}$ and $\{j_{n,m}(x)\}$. Also, we are going to introduce two classes of polynomials: $\{F_{n,m}(x)\}$ and $\{f_{n,m}(x)\}$. For $m = 2$, these polynomials are studied in [4]. Namely, we are going to exhibit some basic properties of the polynomials $\{J_{n,m}(x)\}$, $\{j_{n,m}(x)\}$, $\{F_{n,m}(x)\}$, and $\{f_{n,m}(x)\}$, to generalize the properties of the corresponding polynomials in [4].

2. POLYNOMIALS $J_{n,m}(x)$ AND $j_{n,m}(x)$

Using (1.1) and (1.2), we find the first $m+3$-members of the sequences $\{J_{n,m}(x)\}$ and $\{j_{n,m}(x)\}$, which are given in Table 1.

TABLE 1

n	$J_{n,m}(x)$	$\cdots$	$j_{n,m}(x)$	$\cdots$
0	0	$\cdots$	2	$\cdots$
1	1	$\cdots$	1	$\cdots$
2	1	$\cdots$	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	1	$\cdots$	1	$\cdots$
m	1	$\cdots$	$1+4x$	$\cdots$
$m+1$	$1+2x$	$\cdots$	$1+6x$	$\cdots$
$m+2$	$1+4x$	$\cdots$	$1+8x$	$\cdots$
$m+3$	$1+6x$	$\cdots$	$1+10x$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Using the standard method, we find that the polynomials $\{J_{n,m}(x)\}$ have the following generating function,

$$F(x,t)=(1-t-2xt^m)^{-1}=\sum_{n=1}^{+\infty}J_{n,m}(x)t^{n-1}, \tag{2.1}$$

and the polynomials $\{j_{n,m}(x)\}$ have the generating function

$$G(x,t)=(1+4xt^{m-1})(1-t-2xt^m)^{-1}=\sum_{n=1}^{+\infty}j_{n,m}(x)t^{n-1}. \tag{2.2}$$

From (2.1) and (2.2), we get the following explicit representations:

$$J_{n,m}(x)=\sum_{k=0}^{[(n-1)/m]}\binom{n-1-(m-1)k}{k}(2x)^k; \tag{2.3}$$

$$j_{n,m}(x)=\sum_{k=0}^{[n/m]}\frac{n-(m-2)k}{n-(m-1)k}\binom{n-(m-1)k}{k}(2x)^k. \tag{2.4}$$

For $m=2$ in (2.3) and (2.4), we get the known polynomials $\{J_n(x)\}$ and $\{j_n(x)\}$ (see [4]), respectively.

We can prove the following theorem.

Theorem 2.1: The polynomials $J_{n,m}(x)$ and $j_{n,m}(x)$ satisfy the following equalities, where the superscript (k) denotes the k^{th} derivative with respect to x:

$$j_{n,m}(x)=J_{n,m}(x)+4xJ_{n+1-m,m}(x); \tag{2.5}$$

$$J_{n,m}^{(k)}(x)=J_{n-1,m}^{(k)}(x)+2^k J_{n-m,m}^{(k-1)}(x)+2xJ_{n-m,m}^{(k)}(x), \quad k\ge 1; \tag{2.6}$$

$$j_{n,m}^{(k)}(x)=J_{n,m}^{(k)}(x)+4kJ_{n+1-m,m}^{(k-1)}(x)+4xJ_{n+1-m,m}^{(k)}(x); \tag{2.7}$$

$$j_{n,m}^{(k)}(x)=j_{n-1,m}^{(k)}(x)+2^k j_{n-m,m}^{(k-1)}(x)+2xj_{n-m,m}^{(k)}(x), \quad k\ge 1; \tag{2.8}$$

$$\sum_{i=0}^{n}J_{i,m}^{(k)}(x)J_{n-i,m}^{(s)}(x)=\left(2t^{m-1}(k+s+1)\binom{k+s}{k}\right)^{-1}J_{n,m}^{(k+s+1)}(x); \tag{2.9}$$

$$\sum_{i=0}^{n}J_{i,m}^{(k)}(x)j_{n-i,m}^{(s)}(x)=\frac{2t^{-m}-t^{1-m}}{2(k+s+1)\binom{k+s}{k}}J_{n,m}^{(k+s+1)}(x); \tag{2.10}$$

$$\sum_{i=0}^{n}j_{i,m}^{(k)}(x)j_{n-i,m}^{(s)}(x)=\frac{(2-t)^2}{2t^{m+1}(k+s+1)\binom{k+s}{k}}J_{n,m}^{(k+s+1)}(x); \tag{2.11}$$

$$\sum_{i=1}^{n}J_{i,m}(x)=\frac{J_{n+m,m}(x)-1}{2x}; \tag{2.12}$$

$$\sum_{i=1}^{n}j_{i,m}(x)=\frac{j_{n+m,m}(x)-1}{2x}. \tag{2.13}$$

Proof: From Table 1, we can see that (2.5) is true.

To prove the relations (2.6), (2.7), and (2.8), we will use (1.1), (2.5), and (1.2), respectively. Namely, differentiating (1.1), (2.5), and (1.2) k times with respect to x, we obtain equalities (2.6), (2.7), and (2.8), respectively.

From (2.1), we get

$$\frac{\partial^k F(x,t)}{\partial x^k} = \frac{2^k k!\,t^{mk}}{(1-t-2xt^m)^{k+1}} = \sum_{n=1}^{+\infty} J_{n,m}^{(k)}(x)t^{n-1}. \tag{1}$$

From (2.2), we get

$$\frac{\partial^s G(x,t)}{\partial x^s} = \frac{2^s s!(2-t)t^{ms-1}}{(1-t-2xt^m)^{s+1}} = \sum_{n=1}^{+\infty} j_{n,m}^{(s)}(x)t^{n-1}. \tag{2}$$

Using (1), we obtain

$$\frac{\partial^k F(x,t)}{\partial x^k}\cdot\frac{\partial^s F(x,t)}{\partial x^s} = \frac{2^{k+s}k!\,s!\,t^{m(k+s)}}{(1-t-2xt^m)^{k+s+2}}.$$

Since

$$\frac{\partial^k F(x,t)}{\partial x^k}\cdot\frac{\partial^s F(x,t)}{\partial x^s} = \sum_{n=2}^{\infty}\sum_{i=0}^{n} J_{n-i,m}^{(k)}(x)J_{i,m}^{(s)}(x)t^{n-2},$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{i=0}^{n} J_{n-i,m}^{(k)}(x)J_{i,m}^{(s)}(x)t^{n-1} &= \frac{2^{k+s}k!\,s!\,t^{mk+ms+1}}{(1-t-2xt^m)^{k+s+2}} \quad \text{[by (1)]}\\ &= \frac{2^{k+s+1}(k+s+1)!\,k!\,s!\,t^{m(k+s+1)}}{2(k+s+1)!\,t^{m-1}(1-t-2xt^m)^{k+s+2}}\\ &= \frac{k!\,s!}{2(k+s+1)!\,t^{m-1}}\sum_{n=1}^{\infty} J_{n,m}^{(k+s+1)}(x)t^{n-1}.\end{aligned}$$

By the last equalities, we find

$$\sum_{i=0}^{n} J_{i,m}^{(k)}(x)J_{n-i,m}^{(s)}(x) = \left(2t^{m-1}(k+s+1)\binom{k+s}{k}\right)^{-1} J_{n,m}^{(k+s+1)}(x),$$

which is the desired equality (2.9).

In a similar way, from

$$\frac{\partial^k F(x,t)}{\partial x^k}\cdot\frac{\partial^s G(x,t)}{\partial x^s} = \frac{2^{k+s}k!\,s!(2-t)t^{mk+ms-1}}{(1-t-2xt^m)^{k+s+2}} \quad \text{[by (1) and (2)]},$$

we get (2.10):

$$\sum_{i=0}^{n} J_{i,m}^{(k)}(x)j_{n-i,m}^{(s)}(x) = \frac{2t^{-m}-t^{1-m}}{2(k+s+1)\binom{k+s}{k}} J_{n,m}^{(k+s+1)}(x).$$

Again, from (2), we get the equality (2.11). Using the recurrence relations (1.1) and (1.2), we can prove equalities (2.12) and (2.13), respectively.

Corollary 2.1: For $m=1$, $m=2$, and $m=3$, we obtain (see [4]):

$$\begin{aligned} J_{n,1}(x) &= D_n(x), & j_{n,1}(x) &= d_n(x),\\ J_{n,2}(x) &= J_n(x), & j_{n,2}(x) &= j_n(x),\\ J_{n,3}(x) &= R_n(x), & j_{n,3}(x) &= r_n(x).\end{aligned}$$

Corollary 2.2: For $s=0$ in (2.9) and for $k=0$ in (2.10), we have

$$\sum_{i=0}^{n} J_{i,m}^{(k)}(x)J_{n-i,m}(x) = (2t^{m-1}(k+1))^{-1}J_{n,m}^{(k+1)}(x),$$

and

$$\sum_{i=0}^{n} j_{i,m}^{(s)}(x) J_{n-i,m}(x) = \frac{2t^{-m} - t^{1-m}}{2(s+1)} J_{n,m}^{(s+1)}(x),$$

where $J_{n,m}^{(0)}(x) \equiv J_{n,m}(x)$.

3. POLYNOMIALS $F_{n,m}(x)$ AND $f_{n,m}(x)$

First, we are going to introduce the polynomials $\{F_{n,m}(x)\}$ and $\{f_{n,m}(x)\}$ by

$$F_{n,m}(x) = F_{n-1,m}(x) + 2xF_{n-m,m}(x) + 3, \quad n \geq m, \tag{3.1}$$

with $F_{0,m}(x) = 0$, $F_{n,m}(x) = 1$, $n = 1, 2, \ldots, m-1$, and

$$f_{n,m}(x) = f_{n-1,m}(x) + 2xf_{n-m,m}(x) + 5, \quad n \geq m, \tag{3.2}$$

with $f_{0,m}(x) = 0$, $f_{n,m}(x) = 1$, $n = 1, 2, \ldots, m-1$. So, by (3.1), we find the first $m+2$-members of the sequence $\{F_{n,m}(x)\}$:

$$\begin{aligned} &F_{0,m}(x) = 0, \quad F_{1,m}(x) = 1, \ldots, F_{m-1,m}(x) = 1, \\ &F_{m,m}(x) = 4, \quad F_{m+1,m}(x) = 7 + 2x, \ F_{m+2,m}(x) = 10 + 4x. \end{aligned}$$

By (3.2), we find:

$$\begin{aligned} &f_{0,m}(x) = 0, \quad f_{1,m}(x) = 1, \ldots, f_{m-1,m}(x) = 1, \\ &f_{m,m}(x) = 6, \quad f_{m+1,m}(x) = 11 + 2x, \ f_{m+2,m}(x) = 16 + 4x. \end{aligned}$$

For $m = 2$, the polynomials $\{F_{n,m}(x)\}$ and $\{f_{n,m}(x)\}$ are studied in [4].

Theorem 3.1: The polynomials $F_{n,m}(x)$ and $f_{n,m}(x)$ have, respectively, the following explicit representations:

$$F_{n-1+m,m}(x) = J_{n-1+m,m}(x) + 3\sum_{r=0}^{[n/m]} \binom{n-(m-1)r}{r+1} (2x)^r; \tag{3.3}$$

$$f_{n-1+m,m}(x) = J_{n-1+m,m}(x) + 5\sum_{r=0}^{[n/m]} \binom{n-(m-1)r}{r+1} (2x)^r. \tag{3.4}$$

Proof: From (1.1) and (3.1), we see that (3.3) is true for $n = 1$. Suppose that (3.3) is true for n, i.e.,

$$F_{n-1+m,m}(x) = J_{n-1+m,m}(x) + 3\sum_{r=0}^{[n/m]} \binom{n-(m-1)r}{r+1} (2x)^r.$$

Then

$$\begin{aligned} F_{n+m,m}(x) &= F_{n-1+m,m}(x) + 2xF_{n,m}(x) + 3 \\ &= J_{n-1+m,m}(x) + 3\sum_{r=0}^{[n/m]} \binom{n-(m-1)r}{r+1} (2x)r \\ &\quad + 2x\left(J_{n,m}(x) + 3\sum_{r=0}^{[(n-m+1)/m]} \binom{n+1-m-(m-1)r}{r+1} (2x)^r \right) + 3 \end{aligned}$$

$$= J_{n+m,\,m}(x) + 3\sum_{r=0}^{[(n+1)/m]} \binom{n+1-(m-1)r}{r+1}(2x)^r.$$

By induction on n, we conclude that (3.3) is true for all n.

Similarly, we can prove that equality (3.4) is true for all n.

The polynomials $F_{n,2}(x)$ and $f_{n,2}(x)$ are studied in [4].

Theorem 3.2: The polynomials $\{F_{n,m}(x)\}$ and $\{f_{n,m}(x)\}$ satisfy the following relations:

$$2xF_{n,m}(x) = J_{n+m,\,m}(x) + 2J_{n+1,\,m}(x) - 2x\sum_{i=1}^{m-2} J_{n-i,\,m}(x) - 3; \tag{3.5}$$

$$2xf_{n,m}(x) = J_{n+m,\,m}(x) + 4J_{n+1,\,m}(x) - 2x\sum_{i=1}^{m-2} J_{n-i,\,m}(x) - 5. \tag{3.6}$$

Proof: From (1.1) and (1.2), we see that (3.5) is true for $n = 0, 1, \ldots$. Assume (3.5) is true for $n = k$, i.e.,

$$2xF_{k,m}(x) = J_{k+m,\,m}(x) + 2J_{k+1,\,m}(x) - 2x\sum_{i=1}^{m-2} J_{k-i,\,m}(x) - 3.$$

Then

$$\begin{aligned} F_{k+1,m}(x) &= F_{k,m}(x) + 2xF_{k+1-m,\,m}(x) + 3 \quad \text{[by (3.1)]} \\ &= \frac{J_{k+m,\,m}(x) + 2J_{k+1,\,m}(x) - 2x\sum_{i=1}^{m-2} J_{k-i,\,m}(x) - 3}{2x} \\ &\quad + 2x\frac{J_{k+1,\,m}(x) + 2J_{k+2-m,\,m}(x) - 2x\sum_{i=0}^{m-2} J_{k+1-m-i,\,m}(x) - 3}{2x} + 3 \\ &= \frac{J_{k+1+m,\,m}(x) + 2J_{k+2,\,m}(x) - 2x\sum_{i=0}^{m-2} J_{k+1-i,\,m}(x) - 3}{2x}. \end{aligned}$$

By induction on n, we conclude that (3.5) is true for all n. In a similar way, we can prove that (3.6) is true for all n.

From (3.5) and (3.6), we get

$$f_{n,m}(x) - F_{n,m}(x) = \frac{J_{n+1,\,m}(x) - 1}{x}.$$

For $m = 2$ in the last equality, we obtain the known equality (6.11) in [4].

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