UNIQUENESS OF REPRESENTATIONS BY MORGAN-VOYCE NUMBERS

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1. MORGAN-VOYCE NUMBERS

Consider the recurrence

$$X_{n+2} = 3X_{n+1} - X_n \tag{1.1}$$

with

$$X_0 = a,\ X_1 = b\ \ (a, b \text{ integers}). \tag{1.2}$$

Morgan-Voyce numbers B_n, b_n, and their related numbers C_n, c_n are then generated according to the following scheme in which F_n, L_n symbolize the n^{th} Fibonacci and n^{th} Lucas numbers, respectively:

	X_n	a	b	$X_n = F_m, L_m$
(B)	B_n	0	1	F_{2n}
(b)	b_n	1	1	F_{2n-1}
(C)	C_n	2	3	L_{2n}
(c)	c_n	-1	1	L_{2n-1}

(1.3)

Readers are encouraged to determine the first few members of each of these sequences. In particular, $\{B_n\} = 0, 1, 3, 8, 21, 55, \ldots$.

The sets of numbers (1.3) are special cases of the corresponding sets of polynomials $B_n(x)$, $b_n(x)$, $C_n(x)$, $c_n(x)$ [2] when $x = 1$.

2. REPRESENTATIONS BY B_n

Next, consider the representation of positive integers N by means of B_n:

$$N = \sum_{i=1}^{n} \alpha_i \beta_i \quad (\alpha_i = 0, 1, 2). \tag{2.1}$$

Of special interest is the case as in [3] in which all the α_i in (2.1) are 1, giving rise to the numbers 1, 4, 12, 33, ..., i.e.,

$$\sum_{i=1}^{n} B_i = F_{2n+1} - 1. \tag{2.2}$$

A *minimal representation* is indicated in the abbreviated table (Table 1) in which an empty space signifies 0 (zero). This table has already appeared in [3]. An essential feature of this representation proved in [3] is that no two successive terms in the summation have coefficient 2.

TABLE 1. Minimal Representation for $\{B_n\}$: $n = 1, 2, 3, 4$

N	B_1 = 1	B_2 = 3	B_3 = 8	B_4 = 21	N	B_1 = 1	B_2 = 3	B_3 = 8	B_4 = 21	N	B_1 = 1	B_2 = 3	B_3 = 8	B_4 = 21
1	1				13	2	1	1		24		1		1
2	2				14		2	1		25	1	1		1
3		1			15	1	2	1		26	2	1		1
4	1	1			16			2		27		2		1
5	2	1			17	1		2		28	1	2		1
6		2			18	2		2		29			1	1
7	1	2			19		1	2		30	1		1	1
8			1		20	1	1	2		31	2		1	1
9	1		1		21				1	32		1	1	1
10	2		1		22	1			1	33	1	1	1	1
11		1	1		23	2			1	34	2	1	1	1
12	1	1	1							35		2	1	1

Is this representation unique?

Write S_k for the set of digits 0, 1, 2 of length k in the representation. Let

$$\begin{cases} N_k^{\min} = \text{the smallest integer in } S_k, \\ N_k^{\max} = \text{the largest integer in } S_k, \\ R_k = \text{the range of integers in } S_k, \\ I_k = \text{the number of integers in } S_k. \end{cases} \tag{2.3}$$

Then we readily construct the following scheme (Table 2).

TABLE 2. B_n Representation Summary

k	S_k	R_k	$N_k^{\min}$	$N_k^{\max}$	I_k
1	S_1	1, 2	B_1	B_2-1	$2=F_3$
2	S_2	3, ..., 7	B_2	B_3-1	$5=F_5$
3	S_3	8, ..., 20	B_3	B_4-1	$13=F_7$
4	S_4	21, ..., 54	B_4	B_5-1	$34=F_9$
⋮	⋮	⋮	⋮	⋮	⋮
k	S_k	$F_{2k}, \ldots, F_{2k+2}-1$	$B_k=F_{2k}$	$B_{k+1}-1=F_{2k+2}-1$	F_{2k+1}

Clearly, $I_k = N_k^{\max} - N_k^{\min} + 1 = F_{2k+2} - F_{2k} = F_{2k+1}$.

In each block of length k in Table 1,

$$\begin{cases} \text{the smallest number is necessarily } (0,\ 0,\ 0,\ \ldots,\ 1), \text{ and} \\ \text{the largest number is necessarily } (0,\ 0,\ 0,\ \ldots,\ 2). \end{cases} \tag{2.4}$$

Lemma 1: $B_n \le N \le B_{n+1} - 1$.

E.g., $B_8(=987) \le N = 1000 \le B_9 - 1(=2583)$.

Lemma 2: k is uniquely determined by N.

E.g., $N = 1000 \Rightarrow k = 5$.

Combining the above information, we deduce that

Theorem 1: Every positive integer N has a unique representation of the form

$$N = \sum_{i=1}^{\infty} \alpha_i B_i,$$

where [3] two successive values α_i, α_{i+1} cannot both be 2.

The distinctive pattern fixed in Tables 1 and 2 determines the uniqueness of the representation.

A tabular schedule similar to that in Table 1 (but suppressed here for the sake of brevity) ought now to be constructed for maximal representations by B_n. The embargo on the appearance of two successive coefficients in the summation with the value 2, as in the enunciation of Theorem 1, naturally does not apply for maximality. A fixed pattern of the coefficients emerges in the tabulation of maximal representations for B_n, leading to the conviction that the maximal representation is unique. Where this situation differs from that, say, for Pell numbers [1], is that, while (2.2) in which all coefficients are 1 is there common to both minimal and maximal representations, other summations here are common to both which do not belong to (2.2), e.g., $5 = 2B_1 + B_2$. Also see [3] in this context.

3. OTHER REPRESENTATIONS

(i) C_n (lacunary)

Coming now to the companion number set $\{C_n\} = 2, 3, 7, 18, 47, \ldots$ to $\{B_n\}$, i.e., (1.3)(C), we find that the even tenor of our progress is disrupted. For a start, $C_0 = 2$, $C_1 = 3$, so that there is no possible representation of 1 (unity). Thus, any representation is necessarily *lacunary*. It is no good appealing to C_{-1} as an accommodating adjunct to the set $\{C_n\}$ since $C_{-1} = 3$ (indeed, $C_{-n} = C_n$).

Because of this hiatus, there is also no member in the pattern of the minimal representation of, say, 8 though it can be represented maximally as $8 = 2C_0 + 2C_1$, in which there occur two successive coefficients equal to 2. Except for the lacuna at $N = 1$, the potentially fixed minimal pattern is negated in a regular way at $C_n = 1$, $n \geq 2$. The nature of the representation is therefore hybrid.

(ii) b_n

Turning now to the Morgan-Voyce numbers $\{b_n\}: 1, 2, 5, 13, 34, \ldots$, we encounter a similar set of circumstances to those for $\{B_n\}$. Arguments paralleling those employed in the previous section are likewise applicable to this context. Analogously to Table 1, a *minimal representation* table may be constructed (an entertaining and instructive pastime). As for B_n, the proscription of two successive coefficients equal to 2 in a minimal representation applies here also.

For comparison with the Table 2 Summary for B_n, we here append a Summary (Table 3) for b_n, in which non-capital symbols correspond to the capital symbols specified in (2.3).

TABLE 3. b_n Representation Summary

k	s_k	r_k	$n_k^{\min}$	$n_k^{\max}$	i_k
1	s_1	1	b_1	b_2-1	F_2
2	s_2	$2,\dots,4$	b_2	b_3-1	F_4
3	s_3	$5,\dots,12$	b_3	b_4-1	F_6
4	s_4	$13,\dots,33$	b_4	b_5-1	F_8
⋮	⋮	⋮	⋮	⋮	⋮
k	s_k	$F_{2k-1},\dots,F_{2k+1}-1$	b_k	$b_{k+1}-1$	F_{2k}

Observe that, by (1.3), $i_k=(b_{k+1}-1)-(b_k)+1=b_{k+1}-b_k=F_{2k+1}-F_{2k-1}=F_{2k}$. Uniqueness of the minimal representation is determined by the fixedness of the pattern.

(iii) c_n

Some initial comfort is offered here by the fact that $1=c_1$, $2=2c_1$. But to represent the number 3, we need to revert to the subterfuge of including $-1=c_{-1}$ ($c_{-n}=c_n$ in fact) in our set $\{c_n\}$. This implies that a representation exists which is non-lacunary. There is a purposefulness about the coefficients which then suggests minimality and uniqueness.

4. CONCLUDING OBSERVATIONS

Write

$$\mathcal{B}_n=\sum_{i=1}^{n}B_i\ (2.2),\quad \mathbf{b}_n=\sum_{i=1}^{n}b_i,\quad \mathcal{C}_n=\sum_{i=0}^{n-1}C_i,\quad \mathbf{c}_n=\sum_{i=1}^{n}c_i.$$

Then we discover the following schedule (cf. (1.3)):

	Fibonacci Equivalence	Recurrence Relation
$\mathcal{B}_n$	$F_{2n+1}-1$	$\mathcal{B}_{n+2}=3\mathcal{B}_{n+1}-\mathcal{B}_n-1$
$\mathbf{b}_n$	F_{2n}	$\mathbf{b}_{n+2}=3\mathbf{b}_{n+1}-\mathbf{b}_n$
$\mathcal{C}_n$	$L_{2n+1}-1$	$\mathcal{C}_{n+2}=3\mathcal{C}_{n+1}-\mathcal{C}_n-1$
$\mathbf{c}_n$	$L_{2n}-2$	$\mathbf{c}_{n+2}=3\mathbf{c}_{n+1}-\mathbf{c}_n+2$

Aspects of $\mathcal{B}_n$ and $\mathcal{C}_n$ are discussed in [3], while features of $\mathbf{b}_n$ and $\mathbf{c}_n$ are analyzed in [4].

Peripherally of import to this paper, but also to provide some publicity for the concept, we mention *Brahmagupta polynomials* [5] which relate to $B_n(x)$ and $b_n(x)$ [5], and to $C_n(x)$ and $c_n(x)$ [4]. Historical information on Brahmagupta and his mathematics is given in some detail in [6].

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