COMPLETION OF NUMERICAL VALUES OF GENERALIZED MORGAN-VOYCE AND RELATED POLYNOMIALS

A. F. Horadam
The University of New England, Armidale, Australia 2351
(Submittted August 1998-Final Revision February 1999)

1. MOTIVATION

Two recent publications [2], [3] examined some of the properties of the related polynomial sequences $\{R_n^{(r,u)}(x)\}$ and $\{S_n^{(r,u)}(x)\}$ defined recursively by

$$R_n^{(r,u)}(x) = (x+2)R_{n-1}^{(r,u)}(x) - R_{n-2}^{(r,u)}(x) \quad (n \geq 2), \tag{1.1}$$

$$S_n^{(r,u)}(x) = (x+2)S_{n-1}^{(r,u)}(x) + S_{n-2}^{(r,u)}(x) \quad (n \geq 2). \tag{1.2}$$

with identical initial conditions

$$R_0(x) = u, \quad R_1(x) = x + r + u, \tag{1.3}$$

$$S_0(x) = u, \quad S_1(x) = x + r + u. \tag{1.4}$$

Papers [2] and [3] dealt only with the five values of the subscript pairs, and the notation, indicated immediately below:

$R_n^{(r,u)}(x)$	r	u	$S_n^{(r,u)}(x)$
$xB_n(x)$	0	0	$x\mathcal{B}_n(x)$
$b_{n+1}(x)$	0	1	$\mathbf{c}_{n+1}(x)$
$C_n(x)$	0	2	$\mathcal{C}_n(x)$
$B_{n+1}(x)$	1	1	$\mathcal{B}_{n+1}(x)$
$c_{n+1}(x)$	2	1	$\mathbf{b}_{n+1}(x)$

(1.5)

where $B_n(x)$, $b_n(x)$, $C_n(x)$, and $c_n(x)$ in the R-column are the *Morgan-Voyce polynomials* specified by the following tabulation (a, b being initial conditions for $n = 0, 1$, respectively)

$R_n^{(r,u)}(x)$	a	b
$B_n(x)$	0	1
$b_n(x)$	1	1
$C_n(x)$	2	$2+x$
$c_n(x)$	-1	1

(1.6)

and $\mathcal{B}_n(x)$, $\mathbf{b}_n(x)$, $\mathcal{C}_n(x)$, and $\mathbf{c}_n(x)$ in the S-column are the corresponding polynomials (the *quasi-Morgan-Voyce polynomials*) relating to $S_n^{(r,u)}(x)$.

Let us now examine the consequence of considering the remaining $3^2 - 5 = 4$ superscript pairs

$$(r, u) = (1, 0),\ (2, 0),\ (1, 2),\ (2, 2). \tag{1.7}$$

Readers are encouraged to construct sets of polynomial expressions for $R_n^{(r,u)}(x)$ and $S_n^{(r,u)}(x)$ for the cases listed in (1.7). Particular usage is made of these polynomials when $x = 1$.

Conventions: Write

$$\left.\begin{array}{ll} (i) & R_n^{(r,u)}(1) \equiv R_n^{(r,u)}, \quad \text{so } B_n(1) \equiv B_n, \ldots, \\ (ii) & S_n^{(r,u)}(1) \equiv S_n^{(r,u)}, \quad \text{so } \mathcal{B}_n(1) \equiv \mathcal{B}_n, \ldots \end{array}\right\}. \tag{1.8}$$

Observe that by (1.2), (1.5), and (1.8),

$$\mathcal{B}_n = 3\mathcal{B}_{n-1} + \mathcal{B}_n. \tag{1.9}$$

2. REFERENCE DATA

It is known from [1] that

$$b_n(x) = B_n(x) - B_{n-1}(x), \tag{2.1}$$

$$c_n(x) = B_n(x) + B_{n-1}(x), \tag{2.2}$$

$$C_n(x) = B_{n+1}(x) - B_{n-1}(x), \tag{2.3}$$

while (see [3])

$$\mathbf{b}_n(x) = \mathcal{B}_n(x) + \mathcal{B}_{n+1}(x), \tag{2.4}$$

$$\mathbf{c}_n(x) = \mathcal{B}_n(x) - \mathcal{B}_{n-1}(x), \tag{2.5}$$

$$\mathcal{C}_n(x) = \mathcal{B}_{n+1}(x) + \mathcal{B}_{n-1}(x). \tag{2.6}$$

Moreover (see [1]),

$$B_n = F_{2n}, \tag{2.7}$$

$$b_n = F_{2n-1}, \tag{2.8}$$

$$C_n = L_{2n}, \tag{2.9}$$

$$c_n = L_{2n-1}, \tag{2.10}$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively. For basic information on F_n and L_n, one might consult [4].

Fibonacci and *Lucas polynomials* are defined recursively by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad F_0(x) = 0, \quad F_1(x) = 1; \tag{2.11}$$

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad L_0(x) = 2, \quad L_1(x) = x. \tag{2.12}$$

Particular Cases:

$$x = 1: \quad F_n(1) = F_n, \; L_n(1) = L_n; \tag{2.13}$$

$$x = 2: \quad F_n(2) = P_n, \; L_n(2) = Q_n \tag{2.14}$$

(the n^{th} *Pell* and *Pell-Lucas numbers,* respectively);

$$x = 3: \quad \{F_n(3)\} \equiv \{0, 1, 3, 10, 33, 109, \ldots\} = \{\mathcal{B}_n\}, \tag{2.15}$$

$$\{L_n(3)\} \equiv \{2, 3, 11, 36, 119, 393, \ldots\} = \{\mathcal{C}_n\}, \tag{2.16}$$

as one may readily verify.

Keep in mind the recurrence ($x = 3$ in (2.11))

$$F_n(3) = 3F_{n-1}(3) + F_{n-2}(3). \tag{2.17}$$

Knowledge of the facts from [1]

$$b_n = B_n - B_{n-1}, \tag{2.18}$$

$$c_n = B_n + B_{n-1}, \tag{2.19}$$

and from [3]

$$\mathbf{b}_n = \mathcal{B}_n + \mathcal{B}_{n-1}, \tag{2.20}$$

$$\mathbf{c}_n = \mathcal{B}_n - \mathcal{B}_{n-1}, \tag{2.21}$$

is applicable to the "crossing" correspondence *vis-à-vis* b_n and $\mathbf{c}_n$, and c_n and $\mathbf{b}_n$ in relation to + and – in (2.18)-(2.21), which appears schematically in [3, (4.33)].

Two Useful Theorems:

I. $$R_n^{(r,u)}(x) = P_n^{(r)}(x) + (u-1)b_n(x) \quad [2, \text{Theorem 1}], \tag{2.22}$$

in which

$$P_n^{(0)}(x) = b_{n+1}(x), \tag{2.23}$$

$$P_n^{(1)}(x) = B_{n+1}(x), \tag{2.24}$$

$$P_n^{(2)}(x) = c_{n+1}(x). \tag{2.25}$$

From (2.23)-(2.25) were derived the results for $R_n^{(r,u)}(x)$ in (1.5).

II. $$S_n^{(r,u)}(x) = (x+r+u)\mathcal{B}_n(x) + u\mathcal{B}_{n-1}(x) \quad [3, (4.14)]. \tag{2.26}$$

3. NUMERICAL COMPLETION

A critical elementary question to ask is: Considering the basic property $B_n = R_n^{(0,0)} = F_{2n}$, derived from (1.5), (1.8), and (2.7), what number plays the corresponding role in $\mathcal{B}_n = S_n^{(0,0)}$?

$\mathbf{S_n^{(0,0)}}$

Comparison of (1.9) and (2.17) quickly reveals that

$$S_n^{(0,0)} = F_n(3)\ (= \mathcal{B}_n) \tag{3.1}$$

since both relevant sequences have initial conditions 0, 1 at $n = 0, 1$. Therefore, we would expect $F_n(3) = \mathcal{B}_n$ could effect a role for $S_n^{(r,u)}(x)$ analogous to $F_{2n} = \mathcal{B}_n = R_n^{(0,0)}$ for $R_n^{(r,u)}(x)$. Then it remains for us to discover whether our expectations are fully realized.

$\mathbf{R_n^{(r,u)}}$

Values of $R_n^{(r,u)}$ in (1.5) and (1.8) are known (see [2]), so we need only to enquire into the corresponding situation appropriate to (1.7).

Pairs of values of (r, u) in (1.7) with $x = 1$ now lead by (2.22), (2.24), (2.25), and (2.7)-(2.10), to

$$R_n^{(1,0)} = P_n^{(1)} - b_n = B_{n+1} - b_n = 2F_{2n}, \tag{3.2}$$

$$R_n^{(2,0)} = P_n^{(2)} - b_n = c_{n+1} - b_n = 3F_{2n}, \tag{3.3}$$

$$R_n^{(1,2)} = P_n^{(1)} + b_n = B_{n+1} + b_n = 2F_{2n+1}, \tag{3.4}$$

$$R_n^{(2,2)} = P_n^{(2)} + b_n = c_{n+1} + c_n = F_{2n+3}. \tag{3.5}$$

$\mathbf{S_n^{(r,u)}}$

Pairs of values of (r, u) in (1.5) with $x = 1$ disclose that by (2.26), (3.1), (2.17), and (1.5),

$$S_n^{(0,1)} = 2\mathcal{B}_n + \mathcal{B}_{n-1} = F_{n+1}(3) - F_n(3)\ (= \mathbf{c}_{n+1}), \tag{3.6}$$

$$S_n^{(0,2)} = 3\mathcal{B}_n + 2\mathcal{B}_{n-1} = F_{n+1}(3) + F_{n-1}(3) = L_n(3) = \mathcal{C}_n, \tag{3.7}$$

$$S_n^{(1,1)} = 3\mathcal{B}_n + \mathcal{B}_{n-1} = \mathcal{B}_{n+1} = F_{n+1}(3), \tag{3.8}$$

$$S_n^{(2,1)} = 4\mathcal{B}_n + \mathcal{B}_{n-1} = F_{n+1}(3) + F_n(3) = \mathbf{b}_{n+1}. \tag{3.9}$$

Turning next to (1.7), we determine by (2.26), (3.1), and (2.17) that

$$S_n^{(1,0)} = 2\mathcal{B}_n = 2F_n(3), \tag{3.10}$$

$$S_n^{(2,0)} = 3\mathcal{B}_n = 3F_n(3), \tag{3.11}$$

$$S_n^{(1,2)} = 2(2\mathcal{B}_n + \mathcal{B}_{n-1}) = 2(F_{n+1}(3) - F_n(3)), \tag{3.12}$$

$$S_n^{(2,2)} = 5\mathcal{B}_n + 2\mathcal{B}_{n-1} = 2F_{n+1}(3) - F_n(3). \tag{3.13}$$

Proofs of all the numerical properties stated above are quite straightforward, as the reader may readily verify.

4. SUMMARY AND CONCLUSION

Assembling together all the $2 \times 3^2 = 18$ exhibited superscript values of r, u in $R_n^{(r,u)}$ and $S_n^{(r,u)}$ for convenience and visual comparison, we have the following attractive compact correlation pattern, which thus completes our objective.

TABLE 1. $R_n^{(r,u)}$ **and** $S_n^{(r,u)}$ **for** $r, u = 0, 1, 2$

r, u	$R_n^{(r,u)}$	$S_n^{(r,u)}$
00	$F_{2n}\ (= B_n)$	$F_n(3)\ (= \mathcal{B}_n)$
01	F_{2n+1}	$F_{n+1}(3) - F_n(3)$
02	L_{2n}	$L_n(3)$
11	F_{2n+2}	$F_{n+1}(3)$
21	L_{2n+1}	$F_{n+1}(3) + F_n(3)$
10	$2F_{2n}$	$2F_n(3)$
20	$3F_{2n}$	$3F_n(3)$
12	$2F_{2n+1}$	$2F_{n+1}(3) - 2F_n(3)$
22	F_{2n+3}	$2F_{n+1}(3) - F_n(3)$

Thus, for example,

$$\frac{R_n^{(2,0)}}{R_n^{(1,0)}} = \frac{S_n^{(2,0)}}{S_n^{(1,0)}} = \frac{3}{2}.$$

REFERENCES

1. A. F. Horadam. "New Aspects of Morgan-Voyce Polynomials." In *Applications of Fibonacci Numbers* **7**:161-76. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1998.
2. A. F. Horadam. "A Composite of Morgan-Voyce Generalizations." *The Fibonacci Quarterly* **35.3** (1997):233-39.
3. A. F. Horadam. "Quasi Morgan-Voyce Polynomials and Pell Convolutions." In *Applications of Fibonacci Numbers* **8**:179-93. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1999.
4. S. Vajda. *Fibonacci and Lucas Numbers and the Golden Section: Theory and Applications.* Chichester: Horwood, 1989.

AMS Classification Numbers: 11B39

❖❖❖