# ALTERNATING SUMS OF FOURTH POWERS OF FIBONACCI AND LUCAS NUMBERS 

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## 1. INTRODUCTION

The Fibonacci and Lucas numbers are defined for all integers $n$ as

$$
\begin{cases}F_{n+1}=F_{n}+F_{n-1}, & F_{1}=F_{2}=1 \\ L_{n+1}=L_{n}+L_{n-1}, & L_{1}=1, L_{2}=3\end{cases}
$$

Their Binet forms are $F_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}$ and $L_{n}=\alpha^{n}+\beta^{n}$, where $\alpha$ and $\beta$ are the roots of $x^{2}-x-1=0$.
Inspired by the well-known sum

$$
\begin{equation*}
\sum_{k=1}^{n} F_{k}^{2}=F_{n} F_{n+1} \tag{1.1}
\end{equation*}
$$

Clary and Hemenway [2] obtained factored closed-form expressions for all sums of the form $\sum_{k=1}^{n} F_{m k}^{3}$, where $m$ is an integer. For example, they discovered

$$
\sum_{k=1}^{n} F_{2 k}^{3}= \begin{cases}\frac{1}{4} F_{n}^{2} L_{n+1}^{2} F_{n-1} L_{n+2} & \text { if } n \text { is even }  \tag{1.2}\\ \frac{1}{4} L_{n}^{2} F_{n+1}^{2} L_{n-1} F_{n+2} & \text { if } n \text { is odd }\end{cases}
$$

and

$$
\begin{equation*}
\sum_{k=1}^{n} F_{4 k}^{3}=\frac{1}{8} F_{2 n}^{2} F_{2 n+2}^{2}\left(L_{4 n+2}+6\right) \tag{1.3}
\end{equation*}
$$

Motivated by the results of Clary and Hemenway, we turned to fourth powers to see if similar factorizations could be obtained. In the case of nonalternating sums, we could find nothing to compare with the beautiful formulas of Clary and Hemenway. However, by experimenting with many numerical examples, we found the most interesting results when we considered alternating sums. We present these results in Section 3, and indicate our method of proof in Section 4. As noted in [2], once such identities are discovered, it is usually a comparatively routine matter to prove them. However, to assist us in the proofs, we have discovered a number of striking sums that involve the Lucas numbers, and we present these in Section 2.

## 2. PRELIMINARY RESULTS

We require the following results.

$$
\begin{array}{ll}
F_{n+k}+F_{n-k}=F_{n} L_{k}, & k \text { even }, \\
F_{n+k}+F_{n-k}=L_{n} F_{k}, & k \text { odd, } \\
F_{n+k}-F_{n-k}=F_{n} L_{k}, & k \text { odd, } \tag{2.3}
\end{array}
$$

$$
\begin{array}{ll}
F_{n+k}-F_{n-k}=L_{n} F_{k}, & k \text { even, } \\
L_{n+k}+L_{n-k}=L_{n} L_{k}, & k \text { even, } \\
L_{n+k}+L_{n-k}=5 F_{n} F_{k}, & k \text { odd, } \\
L_{n+k}-L_{n-k}=L_{n} L_{k}, & k \text { odd, } \\
L_{n+k}-L_{n-k}=5 F_{n} F_{k}, & k \text { even, } \\
L_{2 m}-2=L_{m}^{2}, & m \text { odd, } \\
L_{2 m}+2=L_{m}^{2}, & m \text { even, } \\
L_{2 m}+(-1)^{m+1} 2=5 F_{m}^{2} . & \tag{2.11}
\end{array}
$$

Identities (2.1)-(2.8) appear as (5)-(12) in Bergum and Hoggatt [1], while (2.9)-(2.11) can be proved with the aid of the Binet forms for $F_{n}$ and $L_{n}$.

Throughout this paper $m \neq 0$ is an integer. To assist in our proofs, we also make use of four sums which involve Lucas numbers with even subscripts. If $m$ is odd, we have

$$
\sum_{k=1}^{n} L_{2 m k}= \begin{cases}\frac{5 F_{m n} F_{m(n+1)}}{L_{m}}, & n \text { even }  \tag{2.12}\\ \frac{L_{m n} L_{m(n+1)}}{L_{m}}, & n \text { odd }\end{cases}
$$

and

$$
\sum_{k=0}^{n} L_{2 m k}= \begin{cases}\frac{L_{m n} L_{m(n+1)}}{L_{m}}, & n \text { even }  \tag{2.13}\\ \frac{5 F_{m n} F_{m(n+1)}}{L_{m}}, & n \text { odd }\end{cases}
$$

On the right sides of (2.12) and (2.13), the even and odd cases are reversed. Equally surprising, we have found that for $m$ even

$$
\sum_{k=1}^{n}(-1)^{k} L_{2 m k}= \begin{cases}\frac{5 F_{m n} F_{m(n+1)}}{L_{m}}, & n \text { even }  \tag{2.14}\\ -\frac{L_{m n} L_{m(n+1)}}{L_{m}}, & n \text { odd }\end{cases}
$$

and

$$
\sum_{k=0}^{n}(-1)^{k} L_{2 m k}= \begin{cases}\frac{L_{m n} L_{m(n+1)}}{L_{m}}, & n \text { even }  \tag{2.15}\\ -\frac{5 F_{m n} F_{m(n+1)}}{L_{m}}, & n \text { odd. }\end{cases}
$$

The proofs of (2.12)-(2.15) are similar. We illustrate the method by proving (2.13).
Proof of (2.13): Expressing $L_{2 m k}$ in Binet form and summing the resulting geometric progressions, we obtain

$$
\begin{array}{rlr}
\sum_{k=0}^{n} L_{2 m k} & =\frac{\alpha^{2 m n+2 m}-1}{\alpha^{2 m}-1}+\frac{\beta^{2 m n+2 m}-1}{\beta^{2 m}-1} \\
& =\frac{L_{2 m n+2 m}-L_{2 m n}+L_{2 m}-2}{L_{2 m}-2} \\
& =\frac{L_{(2 m n+m)+m}-L_{(2 m n+m)-m}+L_{m}^{2}}{L_{m}^{2}} & {[\mathrm{by}(2.9)]} \\
& =\frac{L_{2 m n+m} L_{m}+L_{m}^{2}}{L_{m}^{2}} & {[\mathrm{by}(2.7)]}  \tag{2.7}\\
& =\frac{L_{(m n+m)+m n}+L_{(m n+m)-m n}}{L_{m}} . &
\end{array}
$$

Since $m$ is odd, (2.13) follows from (2.5) and (2.6).

## 3. THE MAIN RESULTS

We now present our main results. If $m$ is even, then

$$
\begin{align*}
& \sum_{k=1}^{n}(-1)^{k} F_{m k}^{4}=\frac{(-1)^{n} F_{m n} F_{m(n+1)}\left[L_{m} L_{m n} L_{m(n+1)}-4 L_{2 m}\right]}{5 L_{m} L_{2 m}},  \tag{3.1}\\
& \sum_{k=1}^{n}(-1)^{k} L_{m k}^{4}=\frac{5 F_{m n} F_{m(n+1)}\left[L_{m} L_{m n} L_{m(n+1)}+4 L_{2 m}\right]}{L_{m} L_{2 m}}, n \text { even, }  \tag{3.2}\\
& \sum_{k=0}^{n}(-1)^{k} L_{m k}^{4}=-\frac{5 F_{m n} F_{m(n+1)}\left[L_{m} L_{m n} L_{m(n+1)}+4 L_{2 m}\right]}{L_{m} L_{2 m}}, n \text { odd. } \tag{3.3}
\end{align*}
$$

We mention that (3.2) and (3.3) can be combined in a single sum as

$$
\sum_{k=1}^{n}(-1)^{k} L_{m k}^{4}=\frac{(-1)^{n} 5 F_{m n} F_{m(n+1)}\left[L_{m} L_{m n} L_{m(n+1)}+4 L_{2 m}\right]}{L_{m} L_{2 m}}-8\left(1+(-1)^{n+1}\right)
$$

On the other hand, if $m$ is odd, then

$$
\begin{align*}
& \sum_{k=1}^{n}(-1)^{k} F_{m k}^{4}=\frac{(-1)^{n} F_{m n} F_{m(n+1)}\left[L_{m} L_{m n} L_{m(n+1)}+4(-1)^{n+1} L_{2 m}\right]}{5 L_{m} L_{2 m}},  \tag{3.4}\\
& \sum_{k=1}^{n}(-1)^{k} L_{m k}^{4}=\frac{5 F_{m n} F_{m(n+1)}\left[L_{m} L_{m n} L_{m(n+1)}+4 L_{2 m}\right]}{L_{m} L_{2 m}}, \quad n \text { even },  \tag{3.5}\\
& \sum_{k=0}^{n}(-1)^{k} L_{m k}^{4}=-\frac{5 F_{m n} F_{m(n+1)}\left[L_{m} L_{m n} L_{m(n+1)}-4 L_{2 m}\right]}{L_{m} L_{2 m}}, n \text { odd. } \tag{3.6}
\end{align*}
$$

As before, (3.5) and (3.6) can be expressed as a single sum, but we choose to write them separately in order to present the right sides in factored form. This is the reason for the appearance of the zero lower limit.

## 4. THE METHOD OF PROOF

To illustrate the method, we prove (3.4). First, let $n$ be even. In what follows, we note that since $m$ is odd and $\alpha \beta=-1$, then $(\alpha \beta)^{m k}=(-1)^{k}$. Now

$$
\begin{aligned}
\sum_{k=1}^{n}(-1)^{k} F_{m k}^{4} & =\frac{1}{25} \sum_{k=1}^{n}(-1)^{k}\left(\alpha^{m k}-\beta^{m k}\right)^{4} \\
& =\frac{1}{25} \sum_{k=1}^{n}(-1)^{k}\left(L_{4 m k}-4(-1)^{k} L_{2 m k}+6\right) \\
& =\frac{1}{25} \sum_{k=1}^{n}\left((-1)^{k} L_{4 m k}-4 L_{2 m k}+6(-1)^{k}\right) \\
& =\frac{1}{25} \sum_{k=1}^{n}\left((-1)^{k} L_{4 m k}-4 L_{2 m k}\right), \text { since } n \text { is even. }
\end{aligned}
$$

With the use of (2.12) and (2.14), this becomes

$$
\begin{aligned}
\frac{1}{25}\left[\frac{5 F_{2 m n} F_{2 m(n+1)}}{L_{2 m}}-\frac{20 F_{m n} F_{m(n+1)}}{L_{m}}\right] & =\frac{1}{5}\left[\frac{F_{m n} L_{m n} F_{m(n+1)} L_{m(n+1)}}{L_{2 m}}-\frac{4 F_{m n} F_{m(n+1)}}{L_{m}}\right] \\
& =\frac{F_{m n} F_{m(n+1)}\left[L_{m} L_{m n} L_{m(n+1)}-4 L_{2 m}\right]}{5 L_{m} L_{2 m}}
\end{aligned}
$$

If $n$ is odd, then we have

$$
\begin{aligned}
\sum_{k=1}^{n}(-1)^{k} F_{m k}^{4} & =\sum_{k=0}^{n}(-1)^{k} F_{m k}^{4} \quad\left(\text { since } F_{0}=0\right) \\
& =\frac{1}{25} \sum_{k=0}^{n}\left((-1)^{k} L_{4 m k}-4 L_{2 m k}+6(-1)^{k}\right) \\
& =\frac{1}{25} \sum_{k=0}^{n}\left((-1)^{k} L_{4 m k}-4 L_{2 m k}\right), \text { since } n \text { is odd. }
\end{aligned}
$$

With the aid of (2.13) and (2.15), this sum becomes

$$
\begin{aligned}
\frac{1}{25}\left[\frac{-5 F_{2 m n} F_{2 m(n+1)}}{L_{2 m}}-\frac{20 F_{m n} F_{m(n+1)}}{L_{m}}\right] & =-\frac{1}{5}\left[\frac{F_{m n} L_{m n} F_{m(n+1)} L_{m(n+1)}}{L_{2 m}}+\frac{4 F_{m n} F_{m(n+1)}}{L_{m}}\right] \\
& =-\frac{F_{m n} F_{m(n+1)}\left[L_{m} L_{m n} L_{m(n+1)}+4 L_{2 m}\right]}{5 L_{m} L_{2 m}}
\end{aligned}
$$

and this completes the proof.
We remark that the proof of (3.1) is similar since the parities of $n$ must be considered separately, but the proofs of the other results in Section 3 are more straightforward.

## 5. CONCLUSION

During the course of our investigation we discovered two further pairs of sums similar in character to (2.12)-(2.15) which we include here. If $m$ is odd, then

$$
\begin{equation*}
\sum_{k=1}^{n}(-1)^{k} L_{2 m k}=\frac{(-1)^{n} F_{m n} L_{m(n+1)}}{F_{m}} \tag{5.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{k=0}^{n}(-1)^{k} L_{2 m k}=\frac{(-1)^{n} L_{m n} F_{m(n+1)}}{F_{m}} . \tag{5.2}
\end{equation*}
$$

If $m$ is even, then

$$
\begin{equation*}
\sum_{k=1}^{n} L_{2 m k}=\frac{F_{m n} L_{m(n+1)}}{F_{m}}, \tag{5.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{k=0}^{n} L_{2 m k}=\frac{L_{m n} F_{m(n+1)}}{F_{m}} . \tag{5.4}
\end{equation*}
$$

The Lucas counterpart of (1.1), which appears as $\mathrm{I}_{4}$ in [3], is

$$
\begin{equation*}
\sum_{k=1}^{n} L_{k}^{2}=L_{n} L_{n+1}-2=L_{n} L_{n+1}-L_{0} L_{1} . \tag{5.5}
\end{equation*}
$$

The right side of (5.5) suggests the notation $\left[L_{j} L_{j+1}\right]_{0}^{n}$.
We now make an observation about identity (3.4) and its Lucas counterpart. We have found that for $m=1$ they can be expressed as

$$
\begin{equation*}
\sum_{k=1}^{n}(-1)^{k} F_{k}^{4}=\frac{(-1)^{n}}{3} F_{n-2} F_{n} F_{n+1} F_{n+3} \tag{5.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{k=1}^{n}(-1)^{k} L_{k}^{4}=\left[\frac{(-1)^{j}}{3} L_{j-2} L_{j} L_{j+1} L_{j+3}\right]_{0}^{n} . \tag{5.7}
\end{equation*}
$$

They can be proved quite effectively using the method outlined on page 135 of [2]. We illustrate by proving (5.7).

Let $l_{n}$ denote the sum on the left side of (5.7), and let $r_{n}=\frac{(-1)^{n}}{3} L_{n-2} L_{n} L_{n+1} L_{n+3}$. Then

$$
\begin{equation*}
r_{n}-r_{n-1}=\frac{(-1)^{n}}{3} L_{n}\left(L_{n-2} L_{n+1} L_{n+3}+L_{n-3} L_{n-1} L_{n+2}\right) . \tag{5.8}
\end{equation*}
$$

Now, by using the recurrence satisfied by the Lucas numbers, we express $L_{n-2}, L_{n+3}, L_{n-3}, L_{n-1}$, and $L_{n+2}$ in terms of $L_{n}$ and $L_{n+1}$, and substitute in (5.8) to obtain

$$
r_{n}-r_{n-1}=l_{n}-l_{n-1}=\frac{(-1)^{n}}{3} L_{n}^{4} .
$$

Thus, $l_{n}-r_{n}=-r_{0}$, and this proves (5.7).
To conclude, we mention that for $p$ real the sequences $\left\{U_{n}\right\}$ and $\left\{V_{n}\right\}$ defined for all integers $n$ by

$$
\begin{cases}U_{n}=p U_{n-1}+U_{n-2}, & U_{0}=0, U_{1}=1, \\ V_{n}=p V_{n-1}+V_{n-2}, & V_{0}=2, V_{1}=p\end{cases}
$$

generalize the Fibonacci and Lucas numbers, respectively. Identities (2.12)-(2.15), together with the results in Section 3, and (5.1)-(5.4) translate immediately to $U_{n}$ and $V_{n}$. The reason is that if
we replace $F_{n}$ by $U_{n}, L_{n}$ by $V_{n}$, and 5 by $p^{2}+4$, then $U_{n}$ and $V_{n}$ satisfy (2.1)-(2.11), upon which all our proofs are based. For example, if $m$ is odd, (3.4) and (3.5) become, respectively,

$$
\begin{equation*}
\sum_{k=1}^{n}(-1)^{k} U_{m k}^{4}=\frac{(-1)^{n} U_{m n} U_{m(n+1)}\left[V_{m} V_{m n} V_{m(n+1)}+4(-1)^{n+1} V_{2 m}\right]}{\left(p^{2}+4\right) V_{m} V_{2 m}} \tag{5.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{k=1}^{n}(-1)^{k} V_{m k}^{4}=\frac{\left(p^{2}+4\right) U_{m n} U_{m(n+1)}\left[V_{m} V_{m n} V_{m(n+1)}+4 V_{2 m}\right]}{V_{m} V_{2 m}}, n \text { even. } \tag{5.10}
\end{equation*}
$$

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3. V. E. Hoggatt, Jr. Fibonacci and Lucas Numbers. Boston: Houghton Mifflin, 1969; rpt. The Fibonacci Association, Santa Clara, CA, 1979.
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