A REMARK ON PARITY SEQUENCES

James H. Schmerl
Department of Mathematics, University of Connecticut, Storrs, CT 06269
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For an integer $n \geq 2$, let T_n be the unique set of positive integers such that:

(1) $1 \in T_n$;

(2) if $t > 1$, then $t \in T_n$ iff exactly one of $t-1$, $t-n$ is in T_n.

Condition (2) can be rephrased as

The Triple Criterion: If $t \neq 1$, then $|\{t-n, t-1, t\} \cap T_n| \in \{0, 2\}$.

If $n = 2$, then the set T_n is closely related to the Fibonacci sequence; specifically, $t \in T_2$ iff the t^{th} term of the Fibonacci sequence is odd.

We ask, for each n, which numbers are uniquely expressible as the sum of two distinct elements of T_n. In general, for any given n, one can determine exactly which numbers are uniquely expressible. If $n = 2$, it is easy to see that there are five such numbers: $3 = 1+2$, $5 = 1+4$, $7 = 2+5$, $8 = 1+7$, and $10 = 2+8$. If $n = 3$, then there are exactly eight uniquely expressible numbers: $3 = 1+2$, $4 = 1+3$, $5 = 2+3$, $6 = 1+5$, $7 = 2+5$, $8 = 3+5$, $9 = 1+8$, and $16 = 1+15$. If $n = 4$, then there are exactly five uniquely expressible numbers: $3 = 1+2$, $4 = 1+3$, $6 = 2+4$, $8 = 2+6$, and $16 = 4+12$. If $n \geq 3$, then $1, 2, 3 \in T_n$, so that 3 and 4 are uniquely expressible.

The principal theorem of this note answers this question for all other situations. Let U_n be the set of all integers which are uniquely expressible as the sum of two distinct elements of T_n. Thus, we have just observed that

$$U_2 = \{3, 5, 7, 8, 10\},\ U_3 = \{3, 4, 5, 6, 7, 8, 9, 16\}, \text{ and } U_4 = \{3, 4, 6, 8, 16\}.$$

The following principal theorem characterizes U_n for $n \geq 5$.

Theorem: Let $n \geq 5$. Then $U_n = \{3, 4, n^2-n+3, 2n^2-2n+4\}$ if $n = 2^k+1$ for some k, and $U_n = \{3, 4\}$ otherwise.

The remainder of this paper consists of two sections. The first contains a discussion of the motivation for the principal theorem, and the second contains its proof. The second section can be read independently of the first.

1. MOTIVATION

For an integer $n \geq 2$, let $f_1, f_2, f_3, \ldots$ be the sequence defined by the initial conditions

$$f_1 = f_2 = \cdots = f_n = 1$$

and the recurrence relation

$$f_{n+j} = f_j + f_{n+j-1}$$

for $j \geq 1$. If, in particular, $n = 2$, then the Fibonacci sequence has just been defined, and, as another example, if $n = 5$, then we get the sequence

$$1, 1, 1, 1, 1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 34, 45, 60, 80, 106, \ldots.$$

From this sequence, we define another sequence $t_1, t_2, t_3, \ldots,$ which we will call the n^{th} *parity sequence*: we set $t_i = j$ iff the i^{th} odd term in the sequence $f_1, f_2, f_3, \ldots$ is f_j. For example, the 5^{th} parity sequence is

$$1, 2, 3, 4, 5, 7, 9, 12, 13, 17, 22, 23, 24, \ldots.$$

Then $T_n = \{t_1, t_2, t_3, \ldots\}$.

The principal theorem extends the result of [4] but in a somewhat disguised form. What is essentially proved in [4] is this theorem weakened by requiring that n be an even number, thereby eliminating any exceptional cases.

We next discuss some background for the result of [4] and, consequently, of the above theorem. For positive integers $u < v$, the *1-additive sequence based on* u, v is the sequence $s_1, s_2, s_3, \ldots,$ where $s_1 = u$, $s_2 = v$, and s_{n+2} is the least $a > s_{n+1}$ for which there is a unique pair of integers i, j such that $1 \le i < j \le n+1$ and $a = s_i + s_j$. For example, the 1-additive sequence based on 1, 2 is the sequence

$$1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, \ldots,$$

which was introduced by Ulam [5]. This sequence is still not well understood, but it appears to have a quite erratic behavior. Other 1-additive sequences, such as the one based on 2, 3 also exhibit a similar erratic behavior. In contrast to this, the 1-additive sequence based on 2, v, where $v \ge 5$ is an odd number, has a much more predictable behavior.

Finch made the definition in [2] that the (increasing) sequence $s_1, s_2, s_3, \ldots$ is *regular* if there are positive integers m, p, and d such that whenever $i \ge m$, then $s_{i+p} = s_i + d$. (He refers to the least such p as the *period* of the sequence and to the least such d as the *fundamental difference*.) He observed in [2] that a 1-additive sequence having only finitely many even terms is regular. He then went on to make the conjecture, based on extensive numerical evidence, that for relatively prime $u < v$, the 1-additive sequence based on u, v has only finitely many even terms iff one of the following holds:

(i) $u = 2$ and $v \ge 5$ is odd;
(ii) $u = 4$ and $v \ge 5$ is odd;
(iii) $u = 5$ and $v = 6$;
(iv) $u \ge 6$ is even;
(v) $u \ge 7$ is odd and v is even.

For each of the cases (i)-(v), he made a conjecture as to what the finite sets are. For example, in (i) the set of even terms is $\{2, 2v+2\}$, and in (ii) the set is $\{4, 2v+4, 4v+4\}$ provided that $v \ne 2^m - 1$ for any $m \ge 3$. The conjecture for (i) was proved correct in [4], and for (ii) it was proved correct in [1] in the case $v \equiv 1 \pmod 4$. For (iii) the set is

$$\{6, 16, 26, 36, 80, 124, 144, 172, 184, 196, 238, 416, 448\},$$

and in this case the truth of the conjecture can be verified by direct computation.

Now suppose that $D = \{d_1, d_2, \ldots, d_k\}$ is a finite set of integers, where $d_1 < d_2 < \cdots < d_k$. Let us say for now that the sequence $t_1, t_2, t_3, \ldots$ is the *1-incremental sequence based on* D if $t_1 = 1$ and t_{n+1} is the least $a > t_n$ for which there is a unique pair of integers i, j such that $1 \le i \le n$, $1 \le j \le k$, and $a = t_i + d_j$. For example, the 1-incremental sequence based on {1, 5} is

$$1, 2, 3, 4, 5, 7, 9, 12, 13, 17, 22, 23, 24, \ldots.$$

Notice that this sequence is identical to the 5^{th} parity sequence. In general, the n^{th} parity sequence is identical to the 1-incremental sequence based on $\{1, n\}$.

The connection between 1-incremental sequences and the regularity of 1-additive sequences, elaborating on Finch's observation [2], will be discussed next.

Consider the 1-additive sequence $s_1, s_2, s_3, \ldots$ based on u, v, where $u = 2d_1$ is even and v is odd. Suppose that $2d_1, 2d_2, \ldots, 2d_k$ are all the even terms that are no greater than $2(d_{k-1} + d_k)$ occurring in the 1-additive sequence, where $d_1 < d_2 < \cdots < d_k$. Let $t_1, t_2, t_3, \ldots$ be the 1-incremental sequence based on $D = \{d_1, d_2, \ldots, d_k\}$ and let $T = \{t_1, t_2, t_3, \ldots\}$. It is easy to check that

$$\{s_1, s_2, s_3, \ldots\} = \{2t + v - 2 : t \in T\} \cup \{2d_1, 2d_2, \ldots, 2d_k\}.$$

Now consider 1-additive sequences based on $2, v$, where $v \geq 5$ is an odd integer. The result of [4] is thus seen to be equivalent to the principal theorem restricted to even $n \geq 6$. This leads naturally to the question that this theorem answers.

Every n^{th} parity sequence is regular. (In fact, it is obvious that every 1-incremental sequence is regular.) However, even a little more is true for these sequences (and for all 1-incremental sequences based on 2-element sets, as well). Let $P(n)$ be the period of the n^{th} parity sequence $t_1, t_2, t_3, \ldots$, and let $D(n)$ be the fundamental difference. Then, it follows from the Triple Criterion that, for each $i \geq 1$, $t_{i+P(n)} = t_i + D(n)$. Also $D(n)$ is the least $d > 1$ for which none of d, $d-1$, $d-2$, ..., $d-n+2$ is in T_n. Tabulation of $2D(n)$ and $P(n)$ for many even $n \geq 6$ can be found in [3].

2. THE PROOF

We will need an analysis of the $(2^k+1)^{th}$ parity sequence. An analysis of the $(2^k)^{th}$ parity sequence was given in [4]. As a comparison, we summarize that analysis here.

Proposition 1 ([4]): Let $k \geq 1$ and let $n = 2^k$. Let $1 \leq t \leq 4n^2$ and suppose that $t = 2in + j$, where $0 \leq i < 2n$ and $1 \leq j \leq 2n$. Then:

(1) if $i < n$ and $j \leq n$, then $t \in T_{2n}$ iff $in + j \in T_n$;

(2) if $i < n$ and $j > n$, then $t \in T_{2n}$ iff $in + j - n \in T_n$;

(3) if $i \geq n$ and $j \leq n$, then $t \in T_{2n}$ iff $(i-n)n + j \in T_n$ and $j < n$;

(4) if $i \geq n$ and $j > n$, then $t \in T_{2n}$ iff $j = 2n$. □

The following notation from Section 1 will be used. Recall from Section 1 that, for each $n \geq 2$, there is $d \geq 1$ such that, for any $t \geq 1$, $t \in T_n$ iff $t + d \in T_n$. We let $D(n)$ be the least such d. Clearly, $D(n)$ is the least $d \geq 1$ such that $d+1$, $d+2$, $d+3$, ..., $d+n \in T_n$, and also it is the least $d \geq 1$ such that d, $d-1$, $d-2$, ..., $d-(n-2) \notin T_n$.

Using Proposition 1, we can easily prove by induction that, if $n = 2^k$, then the following hold: if $1 \leq i \leq n$, then $in \in T_n$; if $1 \leq j \leq n$, then $(n-1)j \in T_n$; if $i < n$ and $n - i \leq j < n$, then $in + j \notin T_n$. From this it follows that $n^2 - 1$ is the least $d \geq 1$ such that $\{d, d-1, d-2, \ldots, d-n+2\} \cap T_n = \emptyset$. Thus, $D(n) = 4^k - 1 = n^2 - 1$. It can also be shown that $P(n) = 3^k - 1$.

There is another way to characterize the elements of T_{2^k}. We introduce some notation. For nonnegative integers t and i, we let $b_i(t)$ be the i^{th} digit in the binary expansion of t. For example, since $37 = 1 + 4 + 32$, we get that $b_i(37) = 1$ if $i = 0, 2, 5$ and $b_i(37) = 0$ for all other nonnegative integers i.

Proposition 2: Suppose $k \ge 1$ and $n = 2^k$, and let $1 \le t \le n^2 = 2^{2k}$. Then $t \in T_n$ iff whenever $0 \le r < k$, then $b_r(t) \cdot b_{k+r}(t) = 0$.

Proof: Let us first consider the special case of the proposition when $b_{k-1}(t) = 1$, $b_{2k-1}(t) = 0$, and $b_r(t) = 0$ for all $r < k-1$. Clearly, $b_r(t) \cdot b_{k+r}(t) = 0$ for all $r < k$. It is easily checked by induction on k that Proposition 1 implies that all such t are in T_n.

We now turn to the proof of the proposition in general. The proof is by induction on k. For $k = 1$, it is easily checked. Let $n = 2^k$; we will prove it for the case $2n = 2^{k+1}$. Let $1 \le t \le 4n^2$, and (as in Proposition 1) let $t = 2in + j$, where $0 \le i < 2n$ and $1 \le j \le 2n$. The proof splits naturally into the same four cases as does Proposition 1. Since each one is routine, we will do just case (1), where $i < n$ and $j \le n$. Notice that these restrictions on i and j are equivalent to the condition that $b_k(t-1) = b_{2k+1}(t-1) = 0$, and this condition splits into two subcases.

Subcase 1: $b_k(t) = b_{2k+1}(t) = 0$ and $b_r(t) = 1$ for some $r < k$. Since $b_k(t) = 0$, we need only be concerned with $b_r(t) \cdot b_{(k+1)+r}(t)$ for $r < k$. For such r, $b_r(t) = b_r(in + j)$ and $b_{(k+1)+r}(t) = b_{k+r}(in + j)$, so the result easily follows from the inductive hypothesis.

Subcase 2: $b_k(t) = 1$, $b_{2k+1}(t) = 0$, and $b_r(t) = 0$ for all $r < k$. But this is just the special case that was noted at the beginning of the proof. □

In ways analogous to those in Propositions 1 and 2, the sets T_{2^k+1} can be analyzed. This is done in Propositions 3 and 4, respectively.

Proposition 3: Let $k \ge 0$ and let $n = 2^k$. Let $1 \le t \le (2n+1)^2$ and suppose that $t = i(2n+1) + j$, where $0 \le i \le 2n$ and $1 \le j \le 2n+1$. Then:

(1) if $i \le n$ and $j \le n+1$, then $t \in T_{2n+1}$ iff $i(n+1) + j \in T_{n+1}$;

(2) if $i \le n$ and $j > n+1$, then $t \in T_{2n+1}$ iff $i(n+1) + j - n \in T_{n+1}$ and $i \ne n$;

(3) if $i > n$ and $j \le n+1$, then $t \in T_{2n+1}$ iff $(i-n)(n+1) + j \in T_{n+1}$;

(4) if $i > n$ and $j > n+1$, then $t \in T_{2n+1}$ iff $i = 2n$.

Proof: The proof is by induction on k. For $k = 0$, it is easily checked. Consider some $k > 0$, and assume, as the inductive hypothesis, that the proposition holds for all smaller values of k. Let $n = 2^k$, and let $t = i(2n+1) + j$, where $0 \le i \le 2n$ and $1 \le j \le 2n+1$. We proceed by induction on t. The proof splits naturally into four cases. Since each is routine, we will show only case (1), where $i \le n$ and $j \le n+1$. This case splits into three subcases.

Subcase 1: $i = 0$. Then $t = j$, and it is clear that $j \in T_{2n+1}$ and $j \in T_{n+1}$.

Subcase 2: $i > 0$ and $j > 1$. Then, using the Triple Criterion and the inductive hypothesis on t, we see that $t \in T_{2n+1}$ iff

$$t - 1 \in T_{2n+1} \Leftrightarrow t - (2n+1) \notin T_{2n+1}$$

iff

$$i(2n+1) + j - 1 \in T_{2n+1} \Leftrightarrow (i-1)(2n+1) + j \notin T_{2n+1}$$

iff

$$i(n+1) + j - 1 \in T_{n+1} \Leftrightarrow (i-1)(n+1) + j \notin T_{n+1}$$

iff

$$i(n+1) + j \in T_{n+1}.$$

Subcase 3: $i > 0$ and $j = 1$. Then, again using the Triple Criterion and the inductive hypothesis on t, we see that $t \in T_{2n+1}$ iff

$$t-1 \in T_{2n+1} \Leftrightarrow t-(2n+1) \notin T_{2n+1}$$

iff

$$(i-1)(2n+1)+(2n+1) \in T_{2n+1} \Leftrightarrow (i-1)(2n+1)+1 \notin T_{2n+1}$$

iff

$$i(n+1) \in T_{n+1} \Leftrightarrow (i-1)(n+1)+1 \notin T_{n+1}$$

iff

$$i(n+1)+1 \in T_{n+1}. \ \square$$

Proposition 4: Suppose $k \geq 1$ and $n = 2^k$, and let $2 \leq t \leq n^2+1$. Then $t \in T_{n+1}$ iff whenever $0 \leq r < k$, then $b_r(t-2) \geq b_{k+r}(t-2)$.

Proof: The proof is by induction on k. For small values of k, say $k = 1, 2$, it is easily checked. Let $n = 2^k$; we will prove it for the case $2n = 2^{k+1}$. Let $2 \leq t \leq 4n^2+1$, and (as in Proposition 3) let $t = i(2n+1)+j$, where $0 \leq i \leq 2n$ and $1 \leq j \leq 2n+1$. As $t \geq 2$, it is obvious that $2 \leq i+j$. The proof splits naturally into the same four cases as does Proposition 3. Since each one is routine, we will show just case (1), where $i \leq n$ and $j \leq n+1$. Thus, $2 \leq i+j \leq 2n+1 = 2^{k+1}+1$.

Subcase 1: $i+j < 2^k$. Since $t = i2^{k+1}+(i+j)$, where $2 \leq i+j < 2^k$, it is clear that $b_k(t-2) = b_{2k+1}(t-2) = 0$ and also that $b_r(t-2) = b_r(in+(i+j)-2)$ and $b_{k+(r+1)}(t-2) = b_{k+r}(in+(i+j)-2)$ for $r < k$. Therefore, from the inductive hypothesis,

$$t \in T_{2n+1} \Leftrightarrow i(n+1)+j \in T_{n+1} \Leftrightarrow b_r(i(n+1)+j-2) \geq b_{k+r}(i(n+1)+j-2)$$
$$\text{for } r < k \Leftrightarrow b_r(t-2) \geq b_{(k+1)+r}(t-2) \text{ for } r \leq k.$$

Subcase 2: $i+j = 2^k$. Then $b_0(t-2) = b_k(t-2) = b_{2k+1}(t-2) = 0$, and $b_r(t-2) = 1$ if $1 \leq r < k$. Also, $b_{k+1}(t-2) = 0$ iff i is even. Therefore, we have that $b_r(t-2) \geq b_{(k+1)+r}(t-2)$ whenever $0 \leq r \leq k$ iff i is even. On the other hand,

$$t \in T_{2n+1} \Leftrightarrow i(n+1)+j \in T_{n+1} \Leftrightarrow (i+1)n \in T_{n+1} \Leftrightarrow b_k((i+1)n-2) = 0 \Leftrightarrow i \text{ is even.}$$

Subcase 3: $i+j = 2^k+1$. Then $b_k(t-2) = b_{2k+1}(t-2) = 0$ and $b_r(t-2) = 1$ if $0 \leq r < k$. Thus, we have that $b_r(t-2) \geq b_{(k+1)+r}(t-2)$ whenever $0 \leq r \leq k$. On the other hand,

$$i2^{k+1}+2^k+1 \in T_{2n+1} \Leftrightarrow (i+1)n+1 \in T_{n+1},$$

which is the case since $b_r((i+1)n-1) = 1$ for all $r < k$.

Subcase 4: $2^k+2 \leq i+j < 2^{k+1}$. As in Subcase 1, it is clear that $b_k(t-2) = 1$ and also that $b_r(t-2) = b_r(in+(i+j)-2)$ and $b_{k+(r+1)}(t-2) = b_{k+r}(in+(i+j)-2)$ for $r < k$. Therefore, from the inductive hypothesis,

$$t \in T_{2n+1} \Leftrightarrow i(n+1)+j \in T_{n+1} \Leftrightarrow b_r(i(n+1)+j-2) \geq b_{k+r}(i(n+1)+j-2)$$
$$\text{for } r < k \Leftrightarrow b_r(t-2) \geq b_{(k+1)+r}(t-2) \text{ for } r \leq k.$$

Subcase 5: $i+j = 2^{k+1}$. (This subcase is similar to Subcase 2.) Then $b_0(t-2) = b_k(t-2) = 0$ and $b_r(t-2) = 1$ if $1 \leq r < k$. Also, $b_{k+1}(t-2) = 0$ iff i is even. Therefore, we have that $b_r(t-2) \geq b_{(k+1)+r}(t-2)$ whenever $0 \leq r \leq k$ iff i is even. On the other hand,

$$t \in T_{2n+1} \Leftrightarrow i(n+1)+j \in T_{n+1} \Leftrightarrow (i+1)n \in T_{n+1} \Leftrightarrow b_k((i+1)n-2)=0 \Leftrightarrow i \text{ is even.}$$

Subcase 6: $i+j=2^{k+1}+1$. Therefore, we have $i=n$, $j=n+1$, and $t=2^{2k+1}+2^{k+1}+1$. Then $b_r(t-2)=1$ for all $r \le k$. Thus, we have that $b_r(t-2) \ge b_{(k+1)+r}(t-2)$ whenever $0 \le r \le k$. On the other hand,

$$t \in T_{2n+1} \Leftrightarrow n(2n+1)+(n+1) \in T_{2n+1} \Leftrightarrow n(n+1)+(n+1)T_{n+1} \Leftrightarrow 2^{2k}+2^{k+1}+1 \in T_{n+1},$$

which is the case by the inductive hypothesis since $b_r(2^{2k}+2^{k+1}-1)=1$ for all $r<k$. □

Proposition 5: Suppose that $n \ge 2$ and $s=in+j$, where $0 \le i < n$ and $0 \le j < n$. Then:

(1) if $i<n-1$ and $j<n-i-1$, then $D(n)-s \notin T_n$;

(2) if $i<n$ and $j=n-i-1$, then $D(n)-s \in T_n$;

(3) if $i<n-1$ and $j=n-1$, then $D(n)-s \in T_n$.

Proof: The proof is by induction on s. We provide the details. We let $s=in+j$, where $0 \le i < n$ and either $0 \le j \le n-i-1$ or $j=n-1$. Suppose the proposition is true for all smaller values of s. Let $a=D(n)-s$, so a might be negative. We will determine whether or not $a \in T_n$ by seeing whether or not each of $a+n$ and $a+n-1$ is in T_n, and then use the Triple Criterion applied to $\{a, a+n-1, a+n\}$. To do so, it is necessary to know that $a+n \ne 1$. In each case, it will be clear that $a+n \ne 1$ since there will be b such that $a<b<a+n$ and $b \notin T_n$.

Case 1: $i=0$, $0 \le j < n-1$. Then $a+n=n+D(n)-j \in T_n$ since $n-j \in T_n$, and $a+n-1=n+D(n)-j-1 \in T_n$ since $n-j-1 \in T_n$. Therefore, $a \notin T_n$.

Case 2: $i=0$, $j=n-1$. Then $a+n=D(n)+1 \in T_n$ since $1 \in T_n$, and $a+n-1=D(n) \notin T_n$ by the inductive hypothesis. Therefore, $a \in T_n$.

Case 3: $0<i<n-1$, $j=0$. Then $a+n=D(n)=(i-1)n \notin T_n$ and $a+n-1=D(n)-((i-1)n+1) \notin T_n$ by the inductive hypothesis. Therefore, $a \notin T_n$.

Case 4: $i=n-1$, $j=0$. Then $a+n=D(n)-(n-2)n \notin T_n$ and $a+n-1=D(n)-((n-2)n+1) \in T_n$ by the inductive hypothesis. Therefore, $a \in T_n$.

Case 5: $0<i<n-1$, $0<j<n-i-1$. Then $a+n=D(n)-((i-1)n+j) \notin T_n$ and $a+n-1=D(n)-((i-1)n+(j+1)) \notin T_n$ by the inductive hypothesis. Therefore, $a \notin T_n$.

Case 6: $0<i<n-1$, $j=n-i-1$. Then $a+n=D(n)-((i-1)n+j) \notin T_n$ and $a+n-1=D(n)-((i-1)n+(j+1)) \in T_n$ by the inductive hypothesis. Therefore, $a \in T_n$.

Case 7: $0<i<n-1$, $j=n-1$. Then $a+n=D(n)-((i-1)n+(n-1)) \in T_n$ and $a+n-1=D(n)-in \notin T_n$ by the inductive hypothesis. Therefore, $a \in T_n$. □

Two special instances of Proposition 5 will be used later on. If $i=1$, then (2) shows that $D(n)-2n+2 \in T_n$ and (3) shows that $D(n)-2n+1 \in T_n$.

Corollary 6: Let $n \ge 2$.

(1) Then $D(n) \ge n^2-n+1$.

(2) If $n=2^k+1$, then $D(n)=n^2-n+1$.

Proof: It follows from Proposition 5(2) (letting $i=n-1$, $j=0$) that $D(n)-(n-1)n \in T_n$, so that $D(n) \ge n^2-n+1$. For $n=2^k+1$, it follows from Proposition 4 that, if $n^2-n+2 \le t \le n^2+1$, then $t \in T_n$, so that $D(n) \le n^2-n+1$. □

It can be shown that, if $n = 2^k + 1$, then $P(n) = 3^k + 1$.

It follows that, if $n = 2^k + 1$, then $n^2 - n + 2 = 1 + D(n) \in T_n$ and $2n^2 - 2n + 3 = 1 + 2D(n) \in T_n$. We can now deduce a part of the principal theorem.

Corollary 7: Suppose $k \geq 1$ and $n = 2^k + 1$. Let $a, b \in T_n$ be such that $a < b$.

(1) If $a + b = n^2 - n + 3$, then $a = 1$ and $b = n^2 - n + 2$.

(2) If $a + b = 2n^2 - 2n + 4$, then $a = 1$ and $b = 2n^2 - 2n + 3$.

Proof: Let $a, b \in T_n$ such that $a < b$.

(1) Suppose $a + b = n^2 - n + 3$ but $a > 1$. Let $c = a - 2$, $d = b - 2$, and $e = c + d = n^2 - n - 1$. Then $b_{n-1}(e) = 1$ and, for $0 \leq i < 2^k = n - 1$, $b_i(e) = 1$ iff $i < 2^{k-1}$. Since $b \in T_n$ and $b \leq n^2 - n + 1$, it must be that $b \leq n^2 - 2n + 1$, so that $d \leq n^2 - 2n$. Therefore, there is $j < k$ such that $b_{k+j}(c) = 1$ and then, also, $b_j(d) = 1$. Consider some such j. Clearly, for each $i < k$, $b_i(c) \neq b_i(d)$. It is also clear that, if $k \leq i < 2k$, then $b_i(c) = b_i(d)$. But then $1 = b_{k+j}(c) = b_{k+j}(d) = b_j(d) \neq b_j(c)$, contradicting Proposition 4.

(2) Suppose $a + b = 2n^2 - 2n + 4$, but $a > 1$. Then $b \geq n^2 - n + 3$. Let $c = b - (n^2 - n + 1)$, so that $c \geq 2$, $c \in T_n$ by Corollary 6(2), and $a + c = n^2 - n + 3$. It follows from (1) that $a = c$, which is impossible because $a + c$ is odd. □

With the assistance of Proposition 3 or Proposition 4 we can, in general, determine a large initial segment of any n^{th} parity sequence.

Proposition 8: Let $k \geq 0$, $q \geq 1$,, $n = q2^k + 1$, and $m = 2^k + 1$. Suppose $1 \leq t \leq n(m-1)$, and let $t = in + j$, where $0 \leq i < m - 1$ and $1 \leq j \leq n$. Let $j = r2^k + s$, where $0 \leq r < q$ and $1 \leq s \leq m$. Then $t \in T_n$ iff $im + s \in T_m$.

Proof: The proof is a straightforward induction on t. □

Proof of the Theorem: Suppose that $n \geq 5$. As previously observed, $3, 4 \in U_n$. It follows from Corollary 7 that, if $n = 2^k + 1$, then $n^2 - n + 3$ and $2n^2 - 2n + 4$ are in U_n.

For the reverse inclusion, suppose that $a, b \in T_n$ are such that $a < b$, $a + b \geq 5$, and for no $a', b' \in T_n$ is it the case that $a \neq a' < b' \neq b$ and $a' + b' = a + b$.

We can assume that $a + b > 2n$. (For, as is easy to check, if $s \leq 2n$, then the number of pairs $a, b \in T_n$ such that $a < b$ and $a + b = s$ is $\left[\frac{1}{2}(s-1)\right]$ if $s \leq n$, is $\frac{1}{2}(n-1)$ if $s > n$ is odd, is $\frac{n}{2}$ if $s > n$ and n is even, and is $\frac{1}{2}(n-2)$ if $s > n$ is even and n is even.) Since $\{1, 2, 3, \ldots, n\} \subseteq T_n$ and since $\{a + b - 1, a + b - 2, a + b - 3, \ldots, a + b - n\} \cap T_n \neq \emptyset$, it must be that $1 \leq a \leq n$. Also, $b \leq 2D(n) + n$, as other-wise setting $a' = a + D(n)$ and $b' = b - D(n)$ yields a contradiction.

Now let $n = q2^k + 1$, where q is odd. We consider two cases.

$\boldsymbol{a = 1}$: Then $\{b - 1, b - 2, b - 3, \ldots, b - n + 1\} \cap T_n = \emptyset$. Thus, $b = 1 + pD(n)$ for some $p \geq 1$, and also $p \leq 2$, as otherwise $a' = 1 + D(n)$, $b' = 1 + (p-1)D(n)$ would yield a contradiction. By Corollary 7, we can suppose that $q > 1$. Then, from Proposition 8, we get that $2^k + 1 \in T_n$ and, from Proposition 5, that $D(n) - 2^k + 1 \in T_n$. It follows from Corollary 6 that $D(n) - 2^k + 1 > 2^k + 1$. Thus, setting $a' = 2^k + 1$ and $b' = b - 2^k + 1$ yields a contradiction.

$1 < a \leq n$: Then $\{a+b-i : 1 \leq i \leq n \text{ and } i \neq a\} \cap T_n = \emptyset$. Thus, Proposition 5 implies that $b = pD(n) - n + 1$ for some $p > 0$. Either $a+n \in T_n$ or $a+n-1 \in T_n$. Let a' be whichever one is in T_n, and let $b' = b - (a' - a)$. Then, by Proposition 5, $b' \in T_n$, thereby arriving at a contradiction. □

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