IDENTITIES AND CONGRUENCES INVOLVING HIGHER-ORDER EULER-BERNOULLI NUMBERS AND POLYNOMIALS

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1. INTRODUCTION

Let t be a complex number with $|t| < \frac{\pi}{2}$ and let the Euler numbers E_{2n} $(n = 0, 1, 2, \ldots)$ be defined by the coefficients in the expansion of

$$\sec t = \sum_{n=0}^{\infty} E_{2n} \frac{t^{2n}}{(2n)!}.$$

That is, $E_0 = 1$, $E_2 = 1$, $E_4 = 5$, $E_6 = 61$, $E_8 = 1385$, $E_{10} = 50521$, ...

We denote

$$E(n,k) = \sum_{\alpha_1+\alpha_2+\cdots+\alpha_k=n} \frac{E_{2\alpha_1}E_{2\alpha_2}\ldots E_{2\alpha_k}}{(2\alpha_1)!(2\alpha_2)!\ldots(2\alpha_k)!}, \tag{1}$$

where the summation is over all k-dimensional nonnegative integer coordinates $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$ and k is any positive integer. Recently, several researchers have studied the numbers $E(n,k)$. In [3], Wenpeng Zhang obtained an expression for $E(n, 2m+1)$ $(m \geq 1)$ as a linear combination of Euler numbers and obtained some interesting congruence expressions for Euler numbers. The main purpose of this paper is to express $E(n, 2m)$ as a linear combination of second-order Euler numbers, so that some congruence expressions are obtained correspondingly. The two identities,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} B_n^{(pj)} = p^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} B_n^{(j)} \tag{2}$$

and

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} E_n^{(pj)}(0) = p^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} E_n^{(j)}(0), \tag{3}$$

which were obtained by David Zeitlin (see [2], p. 238) are deduced, and some more common results than (2) and (3) are achieved.

2. DEFINITIONS AND LEMMAS

Definition 1: If (A_n) is any sequence with $A_0 = 1$ and if $f(t) = \sum_{n=0}^{\infty} A_n t^n / n!$ is its generating function, then the "umbral" sequence $A_n^{(k)}$ of order k and the associated Appel sequence of polynomials $A_n^{(k)}(x)$ of order k are defined, respectively, by

$$f(t)^k = \sum_{n=0}^{\infty} A_n^{(k)} t^n / n! \tag{4}$$

and

$$e^{xt} f(t)^k = \sum_{n=0}^{\infty} A_n^{(k)}(x) t^n / n!, \tag{5}$$

where k is any integer. Clearly, $A_n^{(k)}(0) = A_n^{(k)}$ and $A_n^{(1)} = A_n$. It is also easy to see that

$$A_n^{(k)}(x) = \sum_{j=0}^{n} \binom{n}{j} A_j^{(k)} x^{n-j} \quad \text{and that} \quad \frac{d}{dx} A_n^{(k)}(x) = n A_{n-1}^{(k)}(x).$$

Remark 1: ***(a)*** When $f(t) = \sec t$, $|t| < \pi/2$, (4) becomes

$$(\sec t)^k = \sum_{n=0}^{\infty} E_n^{(k)} t^n / n!, \tag{6}$$

where $E_n^{(k)}$ are called Euler numbers of order k;

(b) When $f(t) = t/(e^t - 1)$, $|t| < 2\pi$, (4) becomes

$$(t/(e^t - 1))^k = \sum_{n=0}^{\infty} B_n^{(k)} t^n / n!, \tag{7}$$

where $B_n^{(k)}$ are called Bernoulli numbers of order k (cf. [1], [2]);

(c) When $f(t) = 2/(e^t + 1)$, $|t| < \pi$, (5) becomes

$$e^{xt}(2/(e^t + 1))^k = \sum_{n=0}^{\infty} E_n^{(k)}(x) t^n / n!, \tag{8}$$

where $E_n^{(k)}(x)$ are called Euler polynomials of order k (cf. [1], [2]);

(d) When $f(t) = t/(e^t - 1)$, $|t| < 2\pi$, (5) becomes

$$e^{xt}(t/(e^t - 1))^k = \sum_{n=0}^{\infty} B_n^{(k)}(x) t^n / n!, \tag{9}$$

where $B_n^{(k)}(x)$ are called Bernoulli polynomials of order k (cf. [1], [2]).

Clearly, the usual Euler numbers $E_n = E_n^{(1)}$, Bernoulli numbers $B_n = B_n^{(1)}$, Euler polynomials $E_n(x) = E_n^{(1)}(x)$, and Bernoulli polynomials $B_n(x) = B_n^{(1)}(x)$. Using (6), (7), (8), and (9), we have $E_{2n-1}^{(k)} = 0$ $(n \geq 1)$, $E_{2n}^{(k)} = (-1)^n 2^{2n} E_{2n}^{(k)}(\frac{k}{2})$, and $B_n^{(k)} = B_n^{(k)}(0)$.

Definition 2: $\sigma_j(x_1, x_2, \ldots, x_n)$ $(j = 0, 1, 2, \ldots, n)$ are defined as the coefficients of the polynomial

$$(x + x_1)(x + x_2) \cdots (x + x_n) = \sum_{j=0}^{n} \sigma_j(x_1, x_2, \ldots, x_n) x^{n-j}.$$

Lemma 1: $$E_{2n}^{(k)} = \frac{1}{(k-1)(k-2)} E_{2n+2}^{(k-2)} + \frac{k-2}{k-1} E_{2n}^{(k-2)} \quad (k > 2). \tag{10}$$

Proof: By (6), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\frac{1}{(k-1)(k-2)} E_{2n+2}^{(k-2)} + \frac{k-2}{k-1} E_{2n}^{(k-2)} \right) \frac{t^{2n}}{(2n)!} \\
&= \frac{1}{(k-1)(k-2)} \sum_{n=1}^{\infty} E_{2n}^{(k-2)} \frac{t^{2n-2}}{(2n-2)!} + \frac{k-2}{k-1} \sum_{n=0}^{\infty} E_{2n}^{(k-2)} \frac{t^{2n}}{(2n)!} \\
&= \frac{1}{(k-1)(k-2)} \frac{d^2}{dt^2} (\sec t)^{k-2} + \frac{k-2}{k-1} (\sec t)^{k-2} = (\sec t)^k = \sum_{n=0}^{\infty} E_{2n}^{(k)} \frac{t^{2n}}{(2n)!},
\end{aligned} \tag{11}$$

and comparing the coefficient of t^{2n} on both sides of (11), we immediately obtain (10). □

Lemma 2: $E_{2n}^{(2)} = \dfrac{(-1)^n 2^{2n+1}(2^{2n+2}-1)}{n+1} B_{2n+2}.$ (12)

Proof: By (6) and (7), we have

$$\sum_{n=0}^{\infty} \frac{2}{n+1}(2^{2n+2}-1)B_{2n+2}\frac{t^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{2}{n}(2^{2n}-1)B_{2n}\frac{t^{2n-2}}{(2n-2)!}$$

$$= 4\sum_{n=1}^{\infty}(2n-1)(2^{2n}-1)B_{2n}\frac{t^{2n-2}}{(2n)!} = 4\frac{d}{dt}\sum_{n=1}^{\infty}(2^{2n}-1)B_{2n}\frac{t^{2n-1}}{(2n)!}$$

$$= 4\frac{d}{dt}\left(t^{-1}\sum_{n=1}^{\infty}B_{2n}\frac{(2t)^{2n}}{(2n)!} - t^{-1}\sum_{n=1}^{\infty}B_{2n}\frac{t^{2n}}{(2n)!}\right) = 4\frac{d}{dt}\left(t^{-1}\left(\frac{2t}{e^{2t}-1}-1+t\right) - t^{-1}\left(\frac{t}{e^t-1}-1+\frac{1}{2}t\right)\right)$$

$$= \frac{4e^t}{(e^t+1)^2} = \left(\sec\frac{it}{2}\right)^2 = \sum_{n=0}^{\infty} E_{2n}^{(2)}\frac{(\frac{i}{2}t)^{2n}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{2^{2n}}E_{2n}^{(2)}\frac{t^{2n}}{(2n)!}, \tag{13}$$

and comparing the coefficient of t^{2n} on both sides of (13), we immediately obtain (12). □

Remark 2: By (12), we have $E_0^{(2)} = 1$, $E_2^{(2)} = 2$, $E_4^{(2)} = 16$, $E_6^{(2)} = 272$, $E_8^{(2)} = 7936$, $E_{10}^{(2)} = 353792$, $E_{12}^{(2)} = 22368256, \ldots$.

3. MAIN RESULTS

Theorem 1: $E_{2n}^{(2m)} = \dfrac{1}{(2m-1)!}\displaystyle\sum_{j=0}^{m-1}\sigma_{mj}E_{2n+2m-2-2j}^{(2)},$ (14)

where $\sigma_{mj} = \sigma_j(2^2, 4^2, 6^2, \ldots, (2m-2)^2)$, and m is a positive integer.

Proof: We prove Theorem 1 using mathematical induction.

(a) When $m = 1$, (14) is clearly true.

(b) Suppose (14) is true for some natural number m. By the supposition and (10), we have

$$E_{2n}^{(2m+2)} = \frac{1}{(2m+1)(2m)}E_{2n+2}^{(2m)} + \frac{2m}{2m+1}E_{2n}^{(2m)}$$

$$= \frac{1}{(2m+1)!}\sum_{j=0}^{m-1}\sigma_{mj}E_{2n+2m-2j}^{(2)} + \frac{(2m)^2}{(2m+1)!}\sum_{j=0}^{m-1}\sigma_{mj}E_{2n+2m-2j-2}^{(2)}$$

$$= \frac{1}{(2m+1)!}\sum_{j=0}^{m-1}\sigma_{mj}E_{2n+2m-2j}^{(2)} + \frac{(2m)^2}{(2m+1)!}\sum_{j=1}^{m-1}\sigma_{m(j-1)}E_{2n+2m-2j}^{(2)}$$

$$= \frac{1}{(2m+1)!}\left(E_{2n+2m}^{(2)} + \sum_{j=1}^{m-1}(\sigma_{mj} + (2m)^2\sigma_{m(j-1)})E_{2n+2m-2j}^{(2)} + (2m)^2\sigma_{m(m-1)}E_{2n}^{(2)}\right)$$

$$= \frac{1}{(2m+1)!}\left(E_{2n+2m}^{(2)} + \sum_{j=1}^{m-1}\sigma_{(m+1)j}E_{2n+2m-2j}^{(2)} + \sigma_{(m+1)m}E_{2n}^{(2)}\right)$$

$$= \frac{1}{(2m+1)!}\sum_{j=0}^{m}\sigma_{(m+1)j}E_{2n+2m-2j}^{(2)}, \tag{15}$$

and (15) shows that (14) is also true for the natural number $m+1$. From (a) and (b), we know that (14) is true. □

Corollary 1: $E(n, 2m) = \frac{1}{(2m-1)!(2n)!}\sum_{j=0}^{m-1}\sigma_{mj}E^{(2)}_{2n+2m-2-2j}$. (16)

Proof: From formulas (1) and (6), we have

$$\sum_{n=0}^{\infty}E(n, 2m)t^{2n} = \left(\sum_{n=0}^{\infty}E_{2n}\frac{t^{2n}}{(2n)!}\right)^{2m} = (\sec t)^{2m} = \sum_{n=0}^{\infty}E^{(2m)}_{2n}\frac{t^{2n}}{(2n)!}. \tag{17}$$

Comparing the coefficients of t^{2n} on both sides of (17), we have

$$E(n, 2m) = \frac{1}{(2n)!}E^{(2m)}_{2n}. \tag{18}$$

By (14) and (18), we immediately obtain (16). □

Corollary 2: For any odd prime p, we have the congruence

$$E^{(2)}_{p-1} \equiv \begin{cases} 1 \pmod p & \text{if } p \equiv 1 \pmod 4, \\ -1 \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof: Taking $n = 0$ and $2m-1 = p$ in Corollary 1, and noting that $E_0 = E^{(2)}_0 = 1$, $(p-1)! \equiv -1 \pmod p$, we can get

$$0 \equiv p! = \sum_{j=0}^{\frac{p-1}{2}}\sigma_{\frac{p+1}{2}j}E^{(2)}_{p-1-2j} \equiv E^{(2)}_{p-1} + \sigma_{\frac{p+1}{2}\frac{p-1}{2}}E^{(2)}_0$$
$$= E^{(2)}_{p-1} + 2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdots(p-1)^2 \equiv E^{(2)}_{p-1} + (-1)^{\frac{p+1}{2}} \pmod p,$$

where we have used the congruence

$$\sigma_{\frac{p+1}{2}j} \equiv 0 \pmod p,\ j = 1, 2, \ldots, \frac{p-3}{2}.$$

Therefore,

$$E^{(2)}_{p-1} \equiv \begin{cases} 1 \pmod p & \text{if } p \equiv 1 \pmod 4, \\ -1 \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

This completes the proof. □

Corollary 3: For any odd prime p, we have the congruence

$$\frac{2^{p+1}(2^{p+1}-1)}{p+1}B_{p+1} \equiv 1 \pmod p.$$

Proof: By Corollary 2 and (12). □

Remark 3: For $p = 3$, the preceding congruence says that $60B_4 \equiv 1 \pmod 3$ while, for $p > 3$, using Fermat's little theorem, i.e., $2^p \equiv 2 \pmod p$, the congruence says that $12B_{p+1} \equiv 1 \pmod p$. These facts can be derived directly from the standard recursion for Bernoulli numbers.

Theorem 2: If p is any integer, then

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}A_n^{(pj)}(jx)=n!(-x-pA_1)^n, \tag{19}$$

where $A_n^{(k)}(x)$ are defined as in Definition 1.

Proof: We use the notation $[t^n]h(t)$ to denote the coefficient of t^n in the power series expansion at 0 of $h(t)$. Then, by the definition of $A_n^{(k)}(x)$ and the binomial expansion,

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\binom{n}{j}A_n^{(pj)}(jx)&=n![t^n]\sum_{j=0}^{n}(-1)^j\binom{n}{j}e^{tjx}f(t)^{pj}\\&=n![t^n](1-e^{tx}f(t)^p)^n.\end{aligned}$$

If $g(t)=1-e^{tx}f(t)^p$, then $g(0)=0$ and $g'(t)=-e^{tx}(xf(t)^p+pf(t)^{p-1}f'(t))$, so that $g'(0)=-(x+pA_1)$. Thus, $g(t)=-(x+pA_1)t+0(t^2)$ and $g(t)^n=(-x-pA_1)^nt^n+0(t^{n+1})$.

Corollary 4: If p is any integer, then

(a)
$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}E_n^{(pj)}(jx)=n!\left(\frac{p}{2}-x\right)^n; \tag{20}$$

(b)
$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}B_n^{(pj)}(jx)=n!\left(\frac{p}{2}-x\right)^n. \tag{21}$$

Proof: By formula (19), we immediately obtain (20) and (21), since in the Euler case $f(t)=2/(e^t+1)$ and in the Bernoulli case $f(t)=t/(e^t-1)$. In both cases, $A_0=f(0)=1$ and $A_1=f'(0)=-1/2$. □

Corollary 5: If p is any integer, then

(a) $\sum_{j=0}^{n}(-1)^j\binom{n}{j}E_n^{(pj)}(0)=p^n\cdot\frac{n!}{2^n}$,

(b) $\sum_{j=0}^{n}(-1)^j\binom{n}{j}E_n^{(j)}(0)=\frac{n!}{2^n}$,

(c) $\sum_{j=0}^{n}(-1)^j\binom{n}{j}B_n^{(pj)}=p^n\cdot\frac{n!}{2^n}$,

(d) $\sum_{j=0}^{n}(-1)^j\binom{n}{j}B_n^{(j)}=\frac{n!}{2^n}$.

Proof: Taking $x=0$ in Corollary 4, we immediately obtain Corollary 5. □

Remark 4: By Corollary 5, we immediately obtain (2) and (3) (see [2], p. 238).

Corollary 6: $\sum_{j=0}^{2n}(-1)^j\binom{2n}{j}E_{2n}^{(pj)}=0$. (23)

Proof: Taking $x=p/2$ in Corollary 4(a) and noting that $E_{2n}^{(pj)}=(-1)^n2^{2n}E_{2n}^{(pj)}(\frac{pj}{2})$, we immediately obtain (23). □

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