ON THE GENERALIZED LAGUERRE POLYNOMIALS

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1. INTRODUCTION

In this note we shall study two classes of polynomials $\{g_{n,m}^a(x)\}_{n\in N}$ and $\{h_{n,m}^a(x)\}_{n\in N}$. These polynomials are generalizations of Panda's polynomials (see [2], [3]). Also, these polynomials are special cases of the polynomials which were considered in [4] and [5]. For $m=1$, the polynomials $\{g_{n,m}^a(x)\}$ are the well-known Laguerre polynomials $L_n^\alpha(x)$ (see [6]), i.e.,

$$g_{n,1}^a(x) \equiv L_n^{a-1}(x). \tag{1.0}$$

In this paper the polynomials $\{g_{n,m}^a(x)\}$ and $\{h_{n,m}^a(x)\}$ are given by

$$F(x,t) = (1-t^m)^{-a} e^{-\frac{xt}{1-t^m}} = \sum_{n=0}^{\infty} g_{n,m}^a(x) t^n \tag{1.1}$$

and

$$G(x,t) = (1+t^m)^{-a} e^{-\frac{xt}{1+t^m}} = \sum_{n=0}^{\infty} h_{n,m}^a(x) t^n. \tag{1.2}$$

Using (1.1) and (1.2), we shall prove a great number of interesting relations for $\{g_{n,m}^a(x)\}$ and $\{h_{n,m}^a(x)\}$, as well as some mixed relations.

2. RECURRENCE RELATIONS AND EXPLICIT REPRESENTATIONS

First we find two recurrence relations of the polynomials $\{g_{n,m}^a(x)\}$.

Differentiating (1.1) with respect to t, we get

$$\begin{aligned}\frac{\partial F(x,t)}{\partial t} &= (1-t^m)^{-a-1} e^{-\frac{xt}{1-t^m}} (amt^{m-1} - amt^{2m-1} - x - x(m-1)t^m) \\ &= (1-t^m)\sum_{n=1}^{\infty} n g_{n,m}^a(x) t^{n-1}.\end{aligned} \tag{2.1}$$

By (2.1) and from (1.1), we obtain the following recurrence relation:

$$\begin{aligned}&n g_{n,m}^a(x) - (n-m) g_{n-m,m}^a(x) \\ &= am(g_{n-m,m}^{a+1}(x) - g_{n-2m,m}^{a+1}(x)) - x(g_{n-1,m}^{a+1}(x) + (m-1) g_{n-1-m,m}^{a+1}(x)).\end{aligned} \tag{2.2}$$

Again, from (1.1) and (2.1), we get

$$\begin{aligned}n g_{n,m}^a(x) &= -x(g_{n-1,m}^a(x) + (m-1) g_{n-1-m,m}^a(x)) \\ &\quad + (m(a-2)+2n) g_{n-m,m}^a(x) - (m(a-2)+n) g_{n-2m,m}^a(x),\ n \ge 2m.\end{aligned} \tag{2.3}$$

Corollary 2.1: If $m=1$, then (2.2) and (2.3) yield the corresponding relations for Laguerre polynomials:

$$nL_n^{a-1}(x) - (n-1)L_{n-1}^{a-1}(x) = (a-x)L_{n-1}^a(x) - aL_{n-2}^a(x)$$

and

$$nL_n^a(x) = (2n+a-2-x)L_{n-1}^a(x) - (n+a-2)L_{n-2}^a(x), \ n \geq 2.$$

In a similar way, from (1.2), we get the following relations:

$$nh_{n,m}^a(x) = (m-1)xh_{n-1-m,m}^{a+2}(x) - amh_{n-m,m}^{a+1}(x) - xh_{n-1,m}^{a+2}(x), \ n \geq m,$$

and

$$nh_{n,m}^a(x) = x(m-1)h_{n-1-m,m}^a(x) - xh_{n-1,m}^a(x)$$
$$- (2n+am-2m)h_{n-m,m}^a(x) - (n+am-2m)h_{n-2m,m}^a(x), \ n \geq m.$$

Starting from (1.1) and (1.2), we get the following explicit representations of the polynomials $\{g_{n,m}^a(x)\}$ and $\{h_{n,m}^a(x)\}$, respectively:

$$g_{n,m}^a(x) = \sum_{i=0}^{[n/m]} \frac{(-1)^{n-mi}(a+n-mi)_i}{i!(n-mi)!} x^{n-mi} \tag{2.4}$$

and

$$h_{n,m}^a(x) = \sum_{i=0}^{[n/m]} \frac{(-1)^{n-(m-1)i}(a+n-mi)_i}{i!(n-mi)!} x^{n-mi}. \tag{2.5}$$

Corollary 2.2: If $m=1$, then (2.6) is the explicit representation of the Laguerre polynomials:

$$L_n^{a-1}(x) = \sum_{i=0}^{n} \frac{(-1)^{n-i}(a+n-i)_i}{i!(n-i)!} x^{n-i}.$$

Now, differentiating (1.1) with respect to x, we get

$$Dg_{n,m}^a(x) = -g_{n-1,m}^{a+1}(x), \ n \geq 1. \tag{2.6}$$

If we differentiate (2.6), with respect to x, k times, we obtain

$$D^k g_{n,m}^a(x) = (-1)^k g_{n-k,m}^{a+k}(x), \ n \geq k. \tag{2.7}$$

Corollary 2.3: Using the idea in [1], from (2.2) and (2.6), we get

$$(n-xD)g_{n,m}^a(x) = (n-m+x(m-1)D)g_{n-m,m}^a(x) + amD(g_{n+1-2m,m}^a(x) - g_{n+1-m,m}^a(x)).$$

For $m=1$ in the last equality and from (1.0), we get

$$(n+(a-x)D)L_n^{a-1}(x) = (n-1+aD)L_{n-1}^{a-1}(x).$$

In a similar way, from (1.2), we have

$$Dh_{n,m}^a(x) = -h_{n-1,m}^{a+1}(x)$$

and

$$D^s h_{n,m}^a(x) = (-1)^s h_{n-s,m}^{a+s}(x), \ n \geq s.$$

3. SOME IDENTITIES OF THE CONVOLUTION TYPE

In this section we shall prove some interesting identities related to $\{g_{n,m}^a(x)\}$ and $\{h_{n,m}^a(x)\}$. First, from (1.1), we find

$$F(x,t)\cdot F(y,t)=(1-t^m)^{-2a}e^{-\frac{(x+y)t}{1-t^m}}=\sum_{n=0}^{\infty}g_{n,m}^{2a}(x+y)t^n, \tag{3.1}$$

whence we get

$$\sum_{i=0}^{n}g_{n-i,m}^{a}(x)g_{i,m}^{a}(y)=g_{n,m}^{2a}(x+y).$$

Theorem 3.1: The following identities hold:

$$g_{n,m}^{2a}(x)=\sum_{j=0}^{[n/m]}\sum_{i=0}^{n-mj}\frac{y^{n-i-mj}(n-i-mj)_j}{j!(n-i-mj)!}g_{i,m}^{2a}(x+y); \tag{3.2}$$

$$\sum_{i=0}^{n}D^s g_{n-i,m}^{a}(x)D^s g_{i,m}^{a}(y)=g_{n-2s,m}^{2a+2s}(x+y),\ n\geq 2s; \tag{3.3}$$

$$\sum_{i=0}^{n}D^k g_{n-i,m}^{a}(x)D^k h_{i,m}^{a}(x)=g_{n-2k,2m}^{a+k}(2x),\ n\geq 2k; \tag{3.4}$$

$$\sum_{i=0}^{[(n-k)/m]}\frac{(k)_i}{i!}g_{n-k-mi,2m}^{a}(2x)=(-1)^k\sum_{i=0}^{n}g_{n-i-k,m}^{a+k}(x)h_{i,m}^{a}(x); \tag{3.5}$$

$$\sum_{i=0}^{[(n-k)/m]}(-1)^i\frac{(k)_i}{i!}g_{n-k-mi,2m}^{a}(2x)=(-1)^k\sum_{i=0}^{n}h_{n-i-k,m}^{a+k}(x)g_{i,m}^{a}(x); \tag{3.6}$$

$$\sum_{i=0}^{n}g_{n-i,m}^{a}(x)g_{i,m}^{b}(x)=g_{n,m}^{a+b}(2x). \tag{3.7}$$

Proof: From (3.1), we have

$$(1-t^m)^{-2a}e^{-\frac{xt}{1-t^m}}=e^{\frac{yt}{1-t^m}}\sum_{n=0}^{\infty}g_{n,m}^{2a}(x+y)t^n,$$

whence

$$\sum_{n=0}^{\infty}g_{n,m}^{2a}(x)t^n=\left(\sum_{n=0}^{\infty}\frac{y^nt^n}{n!}\right)\left(\sum_{k=0}^{\infty}\binom{-n}{k}(-t^m)^k\right)\left(\sum_{n=0}^{\infty}g_{n,m}^{2a}(x+y)t^n\right).$$

Multiplying the series on the right side, then comparing the coefficients to t^n, by the last equality we get (3.2).

If we differentiate (1.1) s times, with respect to x, we find

$$\frac{\partial^s F(x,t)}{\partial x^s}=(-1)^s t^s(1-t^m)^{-a-s}e^{-\frac{xt}{1-t^m}}. \tag{a}$$

From (a), we get

$$\frac{\partial^s F(x,t)}{\partial x^s}\cdot\frac{\partial^s F(y,t)}{\partial y^s}=\sum_{n=0}^{\infty}g_{n,m}^{2a+2s}(x+y)t^{n+2s}. \tag{i}$$

Since

$$\frac{\partial^s F(x,t)}{\partial x^s}\cdot\frac{\partial^s F(y,t)}{\partial y^s}=\sum_{n=0}^{\infty}\sum_{i=0}^{n}D^s g_{n-i,m}^{a}(x)D^s g_{i,m}^{a}(y)t^n,$$

and, from (i), it follows that

$$\sum_{i=0}^{n} D^s g_{n-i,m}^{a}(x) D^s g_{i,m}^{a}(y) = g_{n-2s,m}^{2a+as}(x+y),\ n \geq 2s.$$

The last identity is the desired identity (3.3).

Differentiating (1.2) k times, with respect to x, we get

$$\frac{\partial^k G(x,t)}{\partial x^k} = (-1)^k t^k (1+t^m)^{-a-k} e^{-\frac{xt}{1+t^m}}. \tag{b}$$

Then, from (a) and (b), we find

$$\frac{\partial^k F(x,t)}{\partial x^k} \cdot \frac{\partial^k G(x,t)}{\partial x^k} = \sum_{n=0}^{\infty} g_{n,2m}^{a+k}(2x) t^{n+2k}. \tag{ii}$$

The left side of (ii) yields

$$\frac{\partial^k F(x,t)}{\partial x^k} \cdot \frac{\partial^k G(x,t)}{\partial x^k} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} D^k g_{n-i,m}^{a}(x) D^k h_{i,m}^{a}(x) t^n. \tag{iii}$$

So, from (ii) and (iii), we get (3.4).

In a similar way, starting from (1.1) and (1.2), we can prove identity (3.5). From (1.1) and (b), we can prove identity (3.6).

In the proof identity (3.7), we start from

$$F^a(x,t) = (1-t^m)^{-a} e^{-\frac{xt}{1-t^m}}, \quad \text{by (1.1)},$$

and

$$F^b(x,t) = (1-t^m)^{-b} e^{-\frac{xt}{1-t^m}}, \quad \text{by (1.1)}.$$

So, we obtain

$$F^a(x,t) \cdot F^b(x,t) = \sum_{n=0}^{\infty} g_{n,m}^{a+b}(2x) t^n.$$

On the other side, we have

$$\left(\sum_{n=0}^{\infty} g_{n,m}^{a}(x) t^n\right)\left(\sum_{n=0}^{\infty} g_{n,m}^{b}(x) t^n\right) = \sum_{n=0}^{\infty} g_{n,m}^{a+b}(2x) t^n.$$

Identity (3.7) follows by the last equality and the proof of Theorem 3.1 is completed.

Corollary 3.1: If $m = 1$ in (3.2), (3.3), and (3.7), then we get

$$L_n^{2a-1}(x) = \sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{y^{n-i-j}(n-i-j)_i}{j!(n-i-j)!} L_i^{2a-1}(x+y),$$

$$\sum_{i=0}^{n} D^s L_{n-i}^{a-1}(x) D^s L_i^{a-1}(y) = L_{n-2s}^{2a+2s-1}(x+y),$$

and

$$\sum_{i=0}^{n} L_{n-i}^{a-1}(x) L_i^{b-1}(x) = L_n^{a+b-1}(2x),$$

respectively.

Furthermore, we shall prove some more general results.

Theorem 3.2:

$$\sum_{i_1+\cdots+i_k=n} g_{i_1,m}^{a_1}(x_1)\cdots g_{i_k,m}^{a_k}(x_k) = g_{n,m}^{a_1+\cdots+a_k}(x_1+\cdots+x_k); \tag{3.8}$$

$$\sum_{i_1+\cdots+i_k=n} h_{i_1,m}^{a_1}(x_1)\cdots h_{i_k,m}^{a_k}(x_k) = h_{n,m}^{a_1+\cdots+a_k}(x_1+\cdots+x_k); \tag{3.9}$$

$$\begin{aligned}&\sum_{s=0}^{n}\sum_{i_1+\cdots+i_k=n-s} g_{i_1,m}^{a}(x_1)\cdots g_{i_k,m}^{a}(x_k)\cdot\sum_{j_1+\cdots+j_k=s} h_{j_1,m}^{a}(x_1)\cdots h_{i_k,m}^{a}(x_k)\\&=\sum_{i_1+\cdots+i_k=n} g_{i_1,2m}^{a}(2x_1)\cdots g_{i_k,2m}^{a}(2x_k).\end{aligned} \tag{3.10}$$

Proof: From (1.1), we get

$$F^{a_1}(x_1,t)\cdots F^{a_k}(x_k,t)=\sum_{n=0}^{\infty} g_{n,m}^{a_1+\cdots+a_k}(x_1+\cdots+x_k)t^n.$$

Further, we have the following identity:

$$\sum_{n=0}^{\infty}\sum_{i_1+\cdots+i_k=n} g_{i_1,m}^{a_1}(x_1)\cdots g_{i_k,m}^{a_k}(x_k)t^n=\sum_{n=0}^{\infty} g_{n,m}^{a_1+\cdots+a_k}(x_1+\cdots+x_k)t^n.$$

Identity (3.8) follows immediately from the last equality. In a similar way, from (1.2), we can prove (3.9).

Now we shall prove (3.10). From (1.1) and (1.2), we have

$$F(x_1,t)\cdots F(x_k,t)\cdot G(x_1,t)\cdots G(x_k,t)=(1-t^{2m})^{-ka}e^{-\frac{2(x_1+\cdots+x_k)t}{1-t^{2m}}}.$$

So we get

$$\left(\sum_{n=0}^{\infty}\sum_{i_1+\cdots+i_k=n} g_{i_1,m}^{a}(x_1)\cdots g_{i_k,m}^{a}(x_k)t^n\right)\cdot\left(\sum_{n=0}^{\infty}\sum_{j_1+\cdots+j_k=n} h_{j_1,m}^{a}(x_1)\cdots h_{j_k,m}^{a}(x_k)t^n\right)$$

$$=\sum_{n=0}^{\infty} g_{n,2m}^{ka}(2x_1+\cdots+2x_k)t^n.$$

Comparing the coefficients to t^n in the last equality, we get (3.10) and the proof of Theorem 3.2 is completed.

Corollary 3.2: If $m=1$, using (1.0), then (3.8) becomes

$$\sum_{i_1+\cdots+i_k=n} L_{i_1}^{a_1-1}(x)\cdots L_{i_k}^{a_k-1}(x)=L_n^{a_1+\cdots+a_k-1}(x_1+\cdots+x_k).$$

Corollary 3.3: If $x_1=x_2=\cdots=x_k=x$ and $a_1=a_2=\cdots=a_k=a$, then (3.8) becomes

$$\sum_{i_1+\cdots+i_k=n} g_{i_1,m}^{a}(x)\cdots g_{i_k,m}^{a}(x)=g_{n,m}^{ka}(kx). \tag{3.11}$$

Corollary 3.4: If $m=1$, then (3.11) yields

$$\sum_{i_1+\cdots+i_k=n} L_{i_1}^{a-1}(x)\cdots L_{i_k}^{a-1}(x)=L_n^{ka-1}(kx).$$

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Errata for "Generalizations of Some Identities Involving the Fibonacci Numbers"
by Fengzhen Zhao & Tianming Wang

The Fibonacci Quarterly **39.2** (2001):165-167

On page 166, (10) should be

$$\sum_{a+b+c=n} U_{ak}U_{bk}U_{ck} = \frac{U_k^2}{2(V_k^2-4q^k)^2}((n-1)(n-2)V_k^2U_{nk}$$
$$-q^kV_k(4n^2-6n-4)U_{(n-1)k}+q^{2k}(4n^2-4)U_{(n-2)k}),\ n\geq 2.$$

Hence, on page 167, (13) should be

$$\sum_{a+b+c=n} F_{ak}F_{bk}F_{ck} = \frac{F_k^2}{2(L_k^2-4(-1)^k)^2}((n-1)(n-2)L_k^2F_{nk}$$
$$-(-1)^kL_k(4n^2-6n-4)F_{(n-1)k}+(4n^2-4)F_{(n-2)k}),\ n\geq 2.$$

In the meantime, line 14 and line 16 of page should be, respectively,

$$\sum_{a+b+c=n} F_{2a}F_{2b}F_{2c} = \frac{1}{50}(9(n-1)(n-2)F_{2n}-3(4n^2-6n-4)F_{2n-2}+(4n^2-4)F_{2n-4}).$$

$$\sum_{a+b+c=n} F_{2a}F_{2b}F_{2c} = \frac{1}{50}((15n^2-63n+66)F_{2n-3}+(10n^2-36n+44)F_{2n-4}).$$

Line 19 of page 167 should be: $+(4n^2-4)P_{(n-2)k}),\ n\geq 2.$
