ANALYTIC CONTINUATION OF THE FIBONACCI DIRICHLET SERIES

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1. INTRODUCTION

Functions defined by Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ are interesting because they often code and link properties of an algebraic nature in analytic terms. This is most often the case when the coefficients a_n are multiplicative arithmetic functions, such as the number or sum of the divisors of n, or group characters. Such series were the first to be studied, and are fundamental in many aspects of number theory. The most famous example of these is undoubtedly $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $(\mathrm{Re}(s) > 1)$, the Riemann zeta function. Initially studied by Euler, who wanted to know the values at the positive integers, it achieved prominence with Riemann, who clarified its intimate connection with the distribution of primes, and gave it lasting notoriety with his hypothesis about the location of its zeros.

Another class of Dirichlet series arises in problems of Diophantine approximation, taking a_n to be the fractional part of $n\theta$, where θ is an irrational number. Their properties depend on how well one can approximate θ by rational numbers, and how these fractional parts are distributed modulo 1. The latter is also a dynamical question about the iterative behavior of the rotation by angle θ of the unit circle. Such functions were defined and studied by Hardy and Littlewood in [3], and also by Hecke [5], Ostrowski and others.

A Dirichlet series typically converges in a half-plane $\mathrm{Re}(s) > \sigma_0$. The first step in retrieving the information contained in it is to study its possible analytic continuation. Even its existence is not usually something that can be deduced immediately from the form of the coefficients, however simple their algebraic or analytic nature may be. For instance, as is well known, $\zeta(s)$ extends meromorphically to the whole complex plane, with only a simple pole at $s = 1$. In addition, it has an important symmetry around $\mathrm{Re}(s) = 1/2$, in the form of a functional equation, a hallmark of many arithmetical Dirichlet series. It has "trivial" zeros at $-2, -4, -6, \ldots$, and its values at the negative odd integers are rational, essentially given by the Bernoulli numbers.

The Diophantine series described above also extend to meromorphic functions on $\mathbb{C}$, but there is no reason to expect a symmetric functional equation. Indeed their poles form the half of a lattice in the left half-plane. Other series, more fancifully defined, are likely not to extend at all. For instance, it is known that $\sum p^{-s}$, where p runs over the primes, cannot extend beyond any point on the imaginary axis, even though it is formed from terms of $\sum_{n=1}^{\infty} n^{-s}$ (Chandrasekharan's book [1] is a nice introduction to these arithmetical connections, whereas Hardy and Riesz's book [4] is a good source for the more analytical aspects of the general theory of Dirichlet series).

The function $\varphi(s)$ we study in this paper, defined by the Dirichlet series $\sum F_n^{-s}$, where F_n is the n^{th} Fibonacci number, shares properties with both types mentioned above. We will show that it extends to a meromorphic function on all of $\mathbb{C}$ and that it has, like the Riemann zeta function, "trivial" zeros at $-2, -6, -10, \ldots$. However, it has trivial simple poles at $0, -4, -8, \ldots$. Again like $\zeta(s)$, we show that at the odd negative integers its values are rational numbers, in this case naturally expressible by Fibonacci and Lucas numbers. In addition, we derive arithmetical expressions for the values of $\varphi(s)$ at positive integers.

On the other hand, we also show that $\varphi(s)$ is analytically similar to the Diophantine series with the golden ratio as the irrational number θ. Indeed $\varphi(s)$ has the same "half-lattice" of poles. More recently, Grabner and Prodinger [2] describe a "Fibonacci" stochastic process in which there arise analytic continuations sharing yet again this kind of pole structure, thus adding a third interesting context in which similar analytic behavior arises. This can be explained by a formal similarity in the calculations in each case, but it would be interesting to study further if there are deeper connections between them.

2. FIRST STEPS

The Fibonacci numbers grow exponentially, and, in general, if $\alpha > 1$ and v_n are integers with $v_n \geq \alpha^n$, then for $\sigma = \operatorname{Re} s > 0$ we have the estimate

$$\sum_{n=1}^{\infty} |v_n^{-s}| \leq \sum_{n=1}^{\infty} v_n^{-\sigma} \leq \sum_{n=1}^{\infty} \alpha^{-\sigma n} = (\alpha^{\sigma} - 1)^{-1}.$$

Hence, the Dirichlet series $\sum_{n=1}^{\infty} v_n^{-s}$ defines an analytic function $f(s)$ for $\sigma > 0$, and furthermore,

$$|sf(s)| \leq |s|(\alpha^{\sigma} - 1)^{-1} = (\log \alpha)^{-1} \frac{|s|}{\sigma} + O(\sigma)$$

as $\sigma \to 0^+$, so that $sf(s)$ is bounded in every angular sector with vertex at 0 opening into the half-plane $\operatorname{Re} s > 0$.

Applying this to the Fibonacci numbers F_n, we get an analytic function $\varphi(s)$ defined for $\sigma = \operatorname{Re} s > 0$ by the Dirichlet series $\sum_{n=1}^{\infty} F_n^{-s}$. It is interesting to express this as a Mellin transform in the classical manner (see Ch. 4 of [4], for example). This is accomplished by the counting function $\Phi(x) = \#\{n \geq 1 : F_n \leq x\}$, which counts the number of Fibonacci numbers less than or equal to x, where we start with F_1 and count $F_1 = F_2 = 1$ twice. Equivalently, $\Phi(x) = \max\{n \geq 0 : F_n \leq x\}$ (but this is not the same as starting from $F_0 = 0$ and counting distinct F_n). Then

$$\varphi(s) = s\int_0^{\infty} \Phi(x) x^{-s-1} dx.$$

Note that $\Phi(x) = 0$ for $0 \leq x < 1$, so the integral actually starts at $x = 1$.

Let $N(x) = [\log_{\varphi} x\sqrt{5}]$, where $\log_{\varphi}$ means the logarithm in base φ and $[x]$ is the integer part of x. Then it is not hard to see that $\Phi(x) = N(x) + \delta(x)$, where $\delta(x) = 0, 1, -1$ and, in fact, $\delta(x) = 1$ if and only if x is in an interval of the form $[F_{2n}, \varphi^{2n}/\sqrt{5})$, $n \geq 1$, and $\delta(x) = -1$ if and only if x is in an interval of the form $[\varphi^{2n+1}/\sqrt{5}, F_{2n+1})$. Let $E \subseteq [1, \infty)$ be the union of these intervals. Then $m(E) < \infty$, where m is Lebesgue measure, and thus we have

$$\varphi(s) = s\int_1^{\infty} \left[\frac{\log(x\sqrt{5})}{\log \varphi}\right] x^{-s-1} dx + s\int_E \delta(x) x^{-s-1} dx. \tag{1}$$

The first integral may be calculated explicitly, and defines a meromorphic function on the whole complex plane. The second integral converges at least for $\sigma > -1$, since for such σ,

$$\int_E |x^{-s-1}| dx = \int_E x^{-\sigma-1} dx < m(E) < \infty.$$

In fact, we do not need to calculate the first integral once we realize that approximating Φ by N is equivalent to approximating F_n by $\varphi^n/\sqrt{5}$ in the Dirichlet series. Indeed,

$$\begin{aligned}\Delta(s) &= \sum_{n=1}^{\infty} \int_{F_{2n}}^{\varphi^{2n}/\sqrt{5}} -dx^{-s} + \int_{\varphi^{2n+1}/\sqrt{5}}^{F_{2n+1}} dx^{-s} \\ &= \sum_{n=1}^{\infty} F_{2n}^{-s} + F_{2n+1}^{-s} - \left(\frac{\varphi^{2n}}{\sqrt{5}}\right)^{-s} - \left(\frac{\varphi^{2n+1}}{\sqrt{5}}\right)^{-s} \\ &= \varphi(s) - 1 - 5^{-s/2}\frac{\varphi^{-2s}}{1-\varphi^{-s}}\end{aligned}$$

for $\sigma > 0$. Thus,

$$\varphi(s) = \Delta(s) + 1 + \frac{\varphi^{cs}}{\varphi^s - 1},$$

where $c = \log_\varphi(\sqrt{5}) - 1$. This is an analytic continuation of $\varphi(s)$ to $\sigma > -1$, and we see that φ has a simple pole at $s = 0$ with residue $1/\log\varphi$. In fact, the series expression

$$\Delta(s) = \sum_{n=2}^{\infty} F_n^{-s} - (\varphi^n/\sqrt{5})^{-s}$$

converges for $\sigma > -2$, since by the mean value theorem,

$$\begin{aligned}|F_n^{-s} - (\varphi^n/\sqrt{5})^{-s}| &\le |F_n - (\varphi^n/\sqrt{5})| \cdot \sup_{|F_n,\, \varphi^n/\sqrt{5}|} |sx^{-s-1}| \\ &= \frac{|s|}{\varphi^n\sqrt{5}} O(\varphi^{-n(\sigma+1)}) = |s| O(\varphi^{-n(\sigma+2)}).\end{aligned}$$

Note also that $\Delta(s) \to 1$ as $|s| \to \infty$ and s lies in an angular sector at 0 opening onto $\operatorname{Re} s > 0$. This is consistent with $\varphi(s) \to 2$ as $|s| \to \infty$ in this manner.

Now we proceed to determine the analytic continuation of $\varphi(s)$ to a meromorphic function on $\mathbb{C}$, and determine its poles. From this we will see the reason for this first "jump" from $\sigma = 0$ to $\sigma = -2$.

3. ANALYTIC CONTINUATION

Proposition 1: The Dirichlet series $\sum_{n=1}^{\infty} F_n^{-s}$ can be continued analytically to a meromorphic function $\varphi(s)$ on $\mathbb{C}$ whose singularities are simple poles at $s = -2k + \frac{\pi i(2n+k)}{\log\varphi}$, $k \ge 0$, $n \in \mathbb{Z}$, with residue $(-1)^k 5^{s/2}\binom{-s}{k}/\log(\varphi)$.

Proof: We obtain the full analytic continuation of $\varphi(s)$ by refining the approximation to F_n to a full binomial series

$$\begin{aligned}F_n^p &= \left(\frac{\varphi^n - \varphi^{*n}}{\sqrt{5}}\right)^p = 5^{-p/2}\varphi^{np}\left(1 - \left(\frac{\varphi^*}{\varphi}\right)^n\right)^p \\ &= 5^{-p/2}\varphi^{np}\left(1 - (-1)^{n+1}\frac{1}{\varphi^{2n}}\right)^p = 5^{-p/2}\sum_{k=0}^{\infty}\binom{p}{k}(-1)^{(n+1)k}\varphi^{n(p-2k)}.\end{aligned} \tag{2}$$

This expansion is valid for any $p \in \mathbb{C}$ and principal powers since then $(xy)^p = x^p y^p$ for $x, y > 0$, and the binomial series converges since $\varphi > 1$. Substituting this into the Dirichlet series for $\varphi(s)$, we get

$$\varphi(s) = \sum_{n=1}^{\infty} F_n^{-s} = 5^{s/2} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{-s}{k} (-1)^{k(n+1)} \varphi^{-n(2+1k)}. \tag{3}$$

The double series (3) is absolutely convergent for $\sigma > 0$, for we have the estimate

$$\begin{aligned} \left|\binom{-s}{k}\right| &= \left|\frac{(-s)(-s-1)\cdots(-s-k+1)}{k!}\right| \\ &\le \left|\frac{|s|(|s|+1)\cdots(|s|+k-1)}{k!}\right| = \binom{|s|+k-1}{k} = (-1)^k \binom{-|s|}{k}, \end{aligned} \tag{4}$$

and then

$$\begin{aligned} \sum_{n\ge1,\, k\ge0}^{\infty} \left|\binom{-s}{k}(-1)^{k(n+1)} \varphi^{-n(s+2k)}\right| &\le \sum_{n\ge1,\, k\ge0}^{\infty} (-1)^k \binom{-|s|}{k} \varphi^{-n(\sigma+2k)} \\ &= \sum_{n=1}^{\infty} \varphi^{n\sigma}(1-\varphi^{-2n})^{-|s|} \le (1-\varphi^{-2})^{-|s|} \sum_{n=1}^{\infty} \varphi^{n\sigma} < \infty. \end{aligned}$$

Interchanging the order of summation, we get, since $|\varphi^{-(s+2k)}| = \varphi^{-(\sigma+2k)} < 1$ for $\sigma > 0, k \ge 0$,

$$\begin{aligned} \varphi(s) &= 5^{s/2} \sum_{k=0}^{\infty} \binom{-s}{k} (-1)^k \sum_{n=1}^{\infty} ((-1)^k \varphi^{-(s+2k)})^n \\ &= 5^{s/2} \sum_{k=0}^{\infty} \binom{-s}{k} (-1)^k \frac{(-1)^k \varphi^{-(s+2k)}}{1-(-1)^k \varphi^{-(s+2k)}} \\ &= 5^{s/2} \sum_{k=0}^{\infty} \binom{-s}{k} \frac{1}{\varphi^{s+2k} + (-1)^{k+1}}. \end{aligned} \tag{5}$$

This series converges uniformly and absolutely on compact subsets of $\mathbb{C}$ not containing any of the poles of the functions

$$f_k(s) = \binom{-s}{k} \frac{1}{\varphi^{s+2k} + (-1)^{k+1}},$$

which are at the points $s = -2k + \frac{\pi i(2n+k)}{\log \varphi}$ for $k \ge 0$ and $n \in \mathbb{Z}$. Thus, they lie on the lines $\sigma = -2k$ spaced at intervals of $\frac{2\pi i}{\log \varphi}$; $s = -2k$ is a pole when k is even, and $s = -2k + \frac{\pi i}{\log \varphi}$ is a pole when k is odd. Here we see the reason for our initial jump from $\sigma = 0$ to $\sigma = -2$. For any $s \in \mathbb{C}$, we have $|\varphi^{s+2k} + (-1)^{k+1}| \ge \varphi^{\sigma+2k} - 1 > \varphi^{\sigma+k}$ for $k \gg 0$; hence,

$$\sum_{k > k_0} |f_k(s)| \le \varphi^{-\sigma} \sum_{k=0}^{\infty} (-1)^k \binom{-|s|}{k} \varphi^{-k} = \varphi^{-\sigma}(1-\varphi^{-1})^{-|s|} < \infty$$

for $k_0 \gg 0$, and this bound is uniform when s varies in a compact set. Hence, (5) defines the analytic continuation of $\varphi(s)$ to a meromorphic function on $\mathbb{C}$ with simple poles at $s_{kn} = -2k + \frac{\pi i(2n+k)}{\log \varphi}$, $k \ge 0$, $n \in \mathbb{Z}$. The residue at s_{kn} is easily seen to be

$$\frac{5^{s_{kn}/2}\binom{-s_{kn}}{k}}{\frac{d}{ds}(\varphi^{s+2k}+(-1)^{k+1})_{s=s_{kn}}}=\frac{(-1)^k 5^{s_{kn}/2}\binom{-s_{kn}}{k}}{\log(\varphi)}. \quad \square \tag{6}$$

4. VALUES AT NEGATIVE INTEGERS

Next, we discuss the values of $\varphi(s)$ at the negative integers. We already know that 0, -4, $-8,\ldots$ are simple poles.

Proposition 2: $\varphi(s)$ has "trivial" zeros at $-m$, where $m\geq 0$, $m\equiv 2 \bmod 4$, and the values at other negative integers are rational numbers, which can be expressed in terms of Fibonacci and Lucas numbers.

Proof: Let $m\geq 0$ be an integer not a multiple of 4. By (5),

$$\varphi(-m)=5^{-m/2}\sum_{k=0}^{\infty}\binom{m}{k}\frac{1}{\varphi^{-m+2k}+(-1)^{k+1}},$$

and since $m\in\mathbb{Z}^+$, all terms with $k>m$ are 0, so that this is really a finite sum belonging to $\mathbb{Q}(\sqrt{5})$. Let $\sigma_k=\binom{m}{k}(\varphi^{-m+2k}+(-1)^{k+1})^{-1}$ and $\alpha_k=\sigma_k+\sigma_{m-k}$, so that $\alpha_k=\alpha_{m-k}$ and

$$\varphi(-1)=\frac{1}{2}5^{-m/2}\sum_{k=0}^{m}\alpha_k,$$

with

$$\varphi(-m)=5^{-m/2}\sum_{k=0}^{\frac{m-1}{2}}\alpha_k$$

if m is odd. We compute

$$\begin{aligned}\alpha_k&=\binom{m}{k}\frac{1}{\varphi^{2k-m}+(-1)^{k+1}}+\binom{m}{m-k}\frac{1}{\varphi^{m-2k}+(-1)^{m-k+1}}\\&=\binom{m}{k}\left(\frac{1}{\varphi^{2k-m}+(-1)^{k+1}}+\frac{1}{(-1)^m\varphi^{*(2k-m)}+(-1)^{m+k+1}}\right)\\&=\binom{m}{k}\left(\frac{1}{\varphi^{2k-m}+(-1)^{k+1}}+\frac{(-1)^m}{\varphi^{*(2k-m)}+(-1)^{k+1}}\right),\end{aligned}$$

so that $\alpha_k^*=\alpha_k$ if m is even, and $\alpha_k^*=-\alpha_k$ if m is odd, where α^* denotes the Galois conjugate in $\mathbb{Q}(\sqrt{5})$. Thus, if m is even, we have $\alpha_k\in\mathbb{Q}$ for all k, and since also $5^{-m/2}\in\mathbb{Q}$, we see that $\varphi(-m)\in\mathbb{Q}$ in this case. If m is odd, then α_k if of the form $a_k\sqrt{5}$, where $a_k\in\mathbb{Q}$, as is also $5^{-m/2}$, so that again $\varphi(-m)\in\mathbb{Q}$. We get further information by carrying through the computation of α_k:

$$\begin{aligned}\alpha_k&=\binom{m}{k}\frac{(-1)^m\varphi^{2k-m}+(-1)^{m+k+1}+\varphi^{*(2k-m)}+(-1)^{k+1}}{(\varphi\varphi^*)^{2k-m}+(-1)^{k+1}(\varphi^{2k-m}+\varphi^*(2k-m)+1)}\\&=\binom{m}{k}(-1)^{k+1}\frac{(-\varphi)^{2k-m}+\varphi^{*(2k-m)}+(-1)^{k+1}(1+(-1)^m)}{\varphi^{2k-m}+\varphi^{*(2k-m)}+(-1)^{k+1}(1+(-1)^m)}.\end{aligned} \tag{7}$$

If $m \equiv 2 \bmod 4$, then this simplifies to $\alpha_k = \binom{m}{k}(-1)^{k+1}$, and then

$$\varphi(-m) = \frac{1}{2}5^{-m/2}\sum_{k=0}^{m}\binom{m}{k}(-1)^{k+1} = 0 \quad (m \geq 0 \text{ even}). \tag{8}$$

These may be considered the "trivial" zeros of $\varphi(s)$.

If m is odd, then

$$\begin{aligned}\alpha_k &= \binom{m}{k}(-1)^{k+1}\frac{-\varphi^{2k-m}+\varphi^{*(2k-m)}}{\varphi^{2k-m}+\varphi^{*(2k-m)}}\\ &= \sqrt{5}\binom{m}{k}(-1)^k\frac{F_{2k-m}}{L_{2k-m}} = \sqrt{5}\binom{m}{k}(-1)^{k+1}\frac{F_{m-2k}}{L_{m-2k}},\end{aligned}$$

where $L_n = \varphi^n + \varphi^{*n}$ is the Lucas sequence $2, 1, 3, 4, 7, \ldots$, and both F_n and L_n are extended to all $n \in \mathbb{Z}$, so that $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$; hence, $F_{-n}/L_{-n} = -F_n/L_n$ for all $n \neq 0$. Then

$$\varphi(-m) = \frac{1}{5^{(m-1)/2}}\sum_{k=0}^{\frac{m-1}{2}}\binom{m}{k}(-1)^{k+1}\frac{F_{m-2k}}{L_{m-2k}} \quad (m \geq 1 \text{ odd}). \;\square \tag{9}$$

All that has been done for the Dirichlet series $\sum_{n=1}^{\infty}F_n^{-s}$ may be carried out in an entirely analogous manner for $\sum_{n=1}^{\infty}(-1)^n F_n^{-s}$. Carrying out the corresponding calculations, which amounts to chasing sign changes in the previous ones, yields the following result.

Theorem 1: The Dirichlet series $\sum_{n=1}^{\infty}(-1)^n F_n^{-s}$ can be analytically continued to a meromorphic function $\psi(s)$ on $\mathbb{C}$ by the series

$$5^{s/2}\sum_{k=0}^{\infty}\binom{-s}{k}\frac{1}{(-1)^{k+1}-\varphi^{s+2k}}. \tag{10}$$

The singularities of $\psi(s)$ are simple poles at

$$s = -2k + \frac{\pi i(2n+k+1)}{\log\varphi}, \quad k \geq 0, \; n \in \mathbb{Z},$$

with residue

$$(-1)^k 5^{s/2}\binom{-s}{k}/\log(\varphi).$$

These are "complementary" to the poles of $\varphi(s)$. Thus, $-m$ is a simple pole for integers $m \geq 0$, $m \equiv 2 \bmod 4$. Similarly, $\psi(s)$ has trivial zeros at $-m$, where $m > 0$, $m \equiv 0 \bmod 4$ (note that $\psi(0) = -1/2$). Finally,

$$\psi(-m) = \varphi(-m) = 5^{-(m-1)/2}\sum_{k=0}^{\frac{m-1}{2}}\binom{m}{k}(-1)^{k+1}\frac{F_{m-2k}}{L_{m-2k}}$$

for $m > 0$, $m \equiv 1 \bmod 2$.

In particular, $\frac{1}{2}(\varphi(s) - \varphi(s))$ analytically continues the series $\sum_{n=0}^{\infty}F_{2n+1}^{-s}$ to a function with simple poles at

$$s = -2k + \frac{\pi i n}{\log\varphi}, \quad k \geq 0, \; n \in \mathbb{Z},$$

hence, at all even negative integers. The odd negative integers are trivial zeros of this function. Similarly, $\frac{1}{2}(\varphi(s)-\varphi(s))$ analytically continues the series $\sum_{n=1}^{\infty} F_{2n}^{-s}$ to a function with the same singularities, and rational values at the odd negative integers.

5. VALUES AT POSITIVE INTEGERS

Theorem 2: For $m \in \mathbb{N}$, $\varphi(m) = 5^{m/2}\sum_{l=1}^{\infty} c_l \varphi^{-l}$, where the coefficients c_l are combinations of sums of powers of divisors of l. In particular, we have the formulas

$$\begin{aligned}
\varphi(1) &= \sqrt{5}\sum_{l=1}^{\infty}(d_1(l)+(-1)^l d_3(l))\,\varphi^{-l},\\
\varphi(2) &= 5\sum_{l\equiv 0 \bmod 2}(-1)^{\frac{l}{2}+1}\sigma_1([l]_2')\,\varphi^{-l},\\
\varphi(3) &= \frac{5\sqrt{5}}{8}\sum_{l=1}^{\infty}(d_3^2(l)-d_3(l)+(-1)^l(d_1^2(l)-d_1(l)))\,\varphi^{-l}\\
\varphi(4) &= \frac{25}{6}\sum_{l\equiv 2 \bmod 4}(\sigma_1([l]_2')-\sigma_3([l]_2'))\,\varphi^{-l}\\
&\quad +25\sum_{l\equiv 0 \bmod 4}\left(\frac{1-[l]_2}{6}\sigma_1([l]_2')+\frac{([l]_2)^3-1}{42}\sigma_3([l]_2'\right)\varphi^{-l},
\end{aligned} \tag{11}$$

where $d_i^k(n) = \sum_{d|n,\, d\equiv i(4)} d^k$, $d_i = d_i^1$, $\sigma_k(n) = \sum_{d|n} d^k$, $[l]_2 = 2^{\mathrm{ord}_2(l)}$ is the 2-part of l, and $[l]_2'$ is the part of l prime to 2.

Proof: Starting from (2) and

$$(-1)^k\binom{-s}{k} = \binom{s+k-1}{k}$$

we have, for $m \in \mathbb{N}$,

$$F_n^{-m} = 5^{m/2}\sum_{k=0}^{\infty}\binom{m+k-1}{m-1}(-1)^{kn}\varphi^{-n(m+2k)}.$$

Let $d = m+2k$, which ranges over $S_m = \{d \ge m : d \equiv m \bmod 2\}$, so

$$F_n^{-m} = 5^{m/2}\sum_{d\in S_m}\binom{\frac{d+m-2}{2}}{m-1}(-1)^{n\frac{d-m}{2}}\varphi^{-nd}.$$

Let $S_m^+ = \{d \ge m : s \equiv m \bmod 4\}$ and $S_m^- = \{d \ge m : s \equiv m+2 \bmod 4\}$. Then

$$F_n^{-m} = 5^{m/2}\left(\sum_{d\in S_m^+}\binom{\frac{d+m-2}{2}}{m-1}\varphi^{-nd} + (-1)^n\sum_{d\in S_m^-}\binom{\frac{d+m-2}{2}}{m-1}\varphi^{-nd}\right).$$

To sum over n, we will collect like powers $l = nd$, so that l runs over all natural numbers and we restrict to $d\,|\,l$, obtaining

$$\sum_{n=1}^{\infty} F_n^{-m} = 5^{m/2}\sum_{l=1}^{\infty}\left(\sum_{d|l,\, d\in S_m^+}\binom{\frac{d+m-2}{2}}{m-1} + \sum_{d|l,\, d\in S_m^-}(-1)^{l/d}\binom{\frac{d+m-2}{2}}{m-1}\right)\varphi^{-l}. \tag{12}$$

Similarly, we may sum over subsets S of the natural numbers, letting l run over multiples of S_m by $n \in S$. For example, if m is odd, then the divisors $d \in S_m$ are odd, so if we wish to sum over odd n, then we let l run over odd numbers. If we wish to sum over even n, then l runs over even numbers. If m is even, then the divisors $d \in S_m$ are even, so l runs over even numbers. In the case $m = 1$, we have $S_1^+ = \{d \geq 1, d \equiv 1 \bmod 4\}$ and $S_1^- = \{d \geq 1, d \equiv 3 \bmod 4\}$. The binomial coefficients reduce to 1. Noting that $\sum_{d|l, d\equiv i\,(4)} (-1)^{l/d} = (-1)^l d_i(l)$, this gives us the formulas:

$$
\begin{aligned}
\sum_{n=1}^{\infty} F_n^{-1} &= \sqrt{5} \sum_{l=1}^{\infty} (d_1(l) + (-1)^l d_3(l))\,\varphi^{-l}; \\
\sum_{n=1}^{\infty} (-1)^n F_n^{-1} &= \sqrt{5} \sum_{l=1}^{\infty} ((-1)^l d_1(l) + d_3(l))\,\varphi^{-l}; \\
\sum_{n=0}^{\infty} F_{2n+1}^{-1} &= \sqrt{5} \sum_{l\equiv 1 \bmod 2} (d_1(l) - d_3(l))\,\varphi^{-l}; \\
\sum_{n=1}^{\infty} F_{2n}^{-1} &= \sqrt{5} \sum_{l\equiv 0 \bmod 2} (d_1(l) + d_3(l))\,\varphi^{-l}.
\end{aligned}
\tag{13}
$$

Horadam [6] treats other approaches to these and other sums of reciprocals ($s = 1$) of quadratic recurrence sequences involving elliptic functions (see Proposition 3 below).

In general,

$$P_m(x) = \binom{\frac{x+m-2}{2}}{m-1}$$

is a polynomial in x of degree $m-1$, divisible by x if m is even. Write

$$P_m(x) = \sum_{k=0}^{m-1} a_{km} x^m,$$

where $a_{km} \in \mathbb{Q}$. Then $\varphi(m) = 5^{m/2} \sum_{l=1}^{\infty} c_l \varphi^{-l}$, where

$$c_l = \sum_{k=0}^{m-1} a_{km} \left(\sum_{d|l,\, d\in S_m^+} d^k + \sum_{k|l,\, d\in S_m^-} (-1)^{l/d} d^k \right). \tag{14}$$

This observation proves the theorem. □

To get the specific formulas for fixed l, m, let s_{klm} denote the expression in parentheses. Note that, for odd m, we have sums over divisor classes $d \equiv 1$, 3 mod 4, so the signs do not bother us:

$$c_l = \sum_{k=0}^{m-1} a_{km} \left(\sum_{d|l,\, d\in S_m^+} d^k + (-1)^l \sum_{k|l,\, d\in S_m^-} d^k \right)$$

and the greater difficulty is the size restriction on divisors, $d \geq m$. For even m, the signs are more of a nuisance. The classes S_m^+, S_m^- are of divisors $d \equiv 0$, 2 mod 4. We are summing over even l, and we write $l = 2^r \lambda$ with $r \geq 1$ and λ odd. Then the divisors $d|l$ with $d \equiv 2 \bmod 4$ are of the form $d = 2\delta$ with $\delta|\lambda$. Thus, forgetting for the moment about the restrictions on the size of d, we note that $\sum_{d|l, d\equiv 2 \bmod 4} d^k = 2^k \sigma_k(\lambda)$ and $\sum_{d|l, d\equiv 2 \bmod 4} (-1)^{l/d} d^k = (-1)^{l/2} 2^k \sigma_k(\lambda)$, since $l/d = \lambda/\delta$ is odd or even according to whether $\lambda = l/2$ is odd or even.

The divisors $d|l$ with $d \equiv 0 \bmod 4$ are nonexistent if $r = 1$; otherwise, they are of the form $d = 2^\rho \delta$, with $2 \le \rho \le r$ and $\delta | \lambda$. Thus,

$$\sum_{d|l,\, d\equiv 2 \bmod 4} d^k = \sum_{\delta|\lambda} \sum_{\rho=2}^{r} (2^\rho \delta)^k = \frac{2^{k(r+1)} - 2^{2k}}{2^k - 1} \sigma_k(\lambda)$$

and

$$\sum_{d|l,\, d\equiv 2 \bmod 4} (-1)^{l/d} d^k = \sum_{\delta|\lambda} \sum_{\rho=2}^{r} (2^\rho \delta)^k (-1)^{2^{r-\rho}\frac{\lambda}{\delta}}$$

$$= -\sum_{\delta|\lambda} 2^{rk}\delta^k + \sum_{\delta|\lambda} \sum_{\rho=2}^{r-1} (2^\rho \delta)^k = -2^{2k}\sigma_k(\lambda).$$

Putting it all together, we obtain the formulas

$$s_{klm} = \begin{cases} 2^k \sigma_k(\lambda) - \displaystyle\sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} d^k, & l \equiv 2 \ (4), \\ (2^k - 2^{2k})\sigma_k(\lambda) - \displaystyle\sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} d^k - \sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} (-1)^{l/d} d^k, & l \equiv 0 \ (4), \end{cases}$$

if $m \equiv 2 \bmod 4$, and

$$s_{klm} = \begin{cases} -2^k \sigma_k(\lambda) + \displaystyle\sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} d^k, & l \equiv 2 \ (4), \\ 2^k \sigma_k(\lambda) + \dfrac{2^{k(r+1)} - 2^{2k}}{2^k - 1}\sigma_k(\lambda) - \displaystyle\sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} d^k - \sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} d^k, & l \equiv 0 \ (4), \end{cases}$$

if $m \equiv 0 \bmod 4$, from which we get the formulas in the theorem.

A curious result may be derived from these formulas in the case $m = 1$, which is probably the subject of Landau's centenary paper [7], to which, unfortunately, the author did not have access. Let $\Theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$ denote Jacobi's theta function. We write also $\Theta(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$, for $q = e^{\pi i z}$ with $\operatorname{Im} z > 0$. Then,

Proposition 3: The following formula holds for $\delta = \frac{i\pi}{\log\varphi}$:

$$\sum_{n\equiv 1 \bmod 2} F_n^{-1} = \frac{\sqrt{5}}{4}\left(\Theta\left(-\frac{1}{\delta}\right)^2 - \Theta\left(-\frac{2}{\delta}\right)^2\right). \tag{15}$$

Proof: Note that $q = \varphi^{-1}$ for $z = -1/\delta$ (δ is the minimum difference of the poles of $\varphi(s)$ along the vertical lines $\sigma = -2k$). We have $\Theta(q)^2 = \sum_{l=0}^{\infty} r_2(l) q^l$, where $r_2(l)$ is the number of representations of l as a sum of two integer squares. Since $r_2(l) = 4(d_1(l) - d_3(l))$, we have shown

$$\sum_{n\equiv 1 \bmod 2} F_n^{-1} = \frac{\sqrt{5}}{4} \sum_{l\equiv 1 \bmod 2} r_2(l)\varphi^{-l},$$

and the formula follows from noting that $r_2(2l) = r_2(l)$ since $d_1(2l) = d_1(l)$, hence

$$\Theta(q^2)^2 = \sum_{l=0}^{\infty} r_2(l)q^{2l} = \sum_{l=0}^{\infty} r_2(2l)q^{2l}$$

and so

$$\sum_{l\equiv 1 \bmod 2} r_2(l)q^l = \Theta(q)^2 - \Theta(q^2)^2. \ \square$$

6. DIOPHANTINE APPROXIMATION

The "Fibonacci zeta function" $\varphi(s)$ has much in common with the meromorphic function obtained by analytic continuation of the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{\{n\varphi\}}{n^s}$ (where $\{x\}$ is the fractional part of x) studied by (among others) Hecke in [5] and Hardy and Littlewood in [3] and related papers. Indeed, they show that this function has the same singularities as $\varphi(s)$, namely, simple poles at $-2k + \frac{(2n+k)\pi i}{\log\varphi}$. They work with any reduced quadratic irrational α, and it is easily seen that we have analogous results in that case also. In particular, $f(s)$ and $\varphi(s)$ differ by an entire function. The function $\varphi(s)$ is not in those papers, which have in mind the study of the distribution of the fractional parts $\{n\alpha\}$ (see also Lang [8]). Hecke mentions that $\sum_{n\in S} n^{-s}$ also has an analytic continuation when S is the set of positive integers satisfying $\{n\alpha\} < \varepsilon$ for a given $\varepsilon > 0$. Note that $F_{2n+1} \in S$ except for finitely many n, but by Weyl's equidistribution theorem there are infinitely more numbers in S, making these continued functions have an additional pole at $s = 1$. Comparing (5) with formulas in [5] and [3], we find similar summands multiplied by zeta-like functions. It would be interesting to obtain more qualitative information. Further questions about $\varphi(s)$ might involve finding nontrivial zeros and studying their distribution, and more properties of the values $\varphi(m)$ at integers m.

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