A DYNAMICAL PROPERTY UNIQUE TO THE LUCAS SEQUENCE

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1. INTRODUCTION

A *dynamical system* is taken here to mean a homeomorphism

$$f : X \to X$$

of a compact metric space X (though the observations here apply equally well to any bijection on a set). The number of points with period n under f is

$$\mathrm{Per}_n(f) = \#\{x \in X \mid f^n x = x\},$$

and the number of points with least period n under f is

$$\mathrm{LPer}_n(f) = \#\{x \in X \mid \#\{f^k x\}_{k \in \mathbb{Z}} = n\}.$$

There are two basic properties that the resulting sequences $(\mathrm{Per}_n(f))$ and $(\mathrm{LPer}_n(f))$ must satisfy if they are finite. First, the set of points with period n is the disjoint union of the sets of points with least period d for each divisor d of n, so

$$\mathrm{Per}_n(f) = \sum_{d|n} \mathrm{LPer}_d(f). \tag{1}$$

Second, if x is a point with least period d, then the d distinct points $x, f(x), f^2(x), \ldots, f^{d-1}(x)$ are all points with least period d, so

$$0 \le \mathrm{LPer}_d(f) \equiv 0 \mod d. \tag{2}$$

Equation (1) may be inverted via the Möbius inversion formula to give

$$\mathrm{LPer}_n(f) = \sum_{d|n} \mu(n/d)\mathrm{Per}_d(f),$$

where $\mu(\cdot)$ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ has a squared factor, and} \\ (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes.} \end{cases}$$

A short proof of the inversion formula may be found in Section 2.6 of [6].

Equation (2) therefore implies that

$$0 \le \sum_{d|n} \mu(n/d)\mathrm{Per}_d(f) \equiv 0 \mod n. \tag{3}$$

Indeed, (3) is the only condition on periodic points in dynamical systems: define a given sequence of nonnegative integers (U_n) to be *exactly realizable* if there is a dynamical system $f : X \to X$ with $U_n = \mathrm{Per}_n(f)$ for all $n \ge 1$. Then (U_n) is exactly realizable if and only if

$$0 \le \sum_{d|n} \mu(n/d)U_d \equiv 0 \mod n \text{ for all } n \ge 1,$$

since the realizing map may be constructed as an infinite permutation using the quantities

$$\frac{1}{n}\sum_{d|n}\mu(n/d)U_d$$

to determine the number of cycles of length n.

Our purpose here is to study sequences of the form

$$U_{n+2}=U_{n+1}+U_n,\ n\geq 1,\ \ U_1=a,\ U_2=b,\ \ a,b\geq 0 \tag{4}$$

with the distinguished Fibonacci sequence denoted (F_n), so

$$U_n=aF_{n-2}+bF_{n-1}\ \text{ for } n\geq 3. \tag{5}$$

Theorem 1: The sequence (U_n) defined by (4) is exactly realizable if and only if $b=3a$.

This result has two parts: the *existence* of the realizing dynamical system is described first, which gives many modular corollaries concerning the Fibonacci numbers. One of these is used later on in the *obstruction* part of the result. The realizing system is (essentially) a very familiar and well-known system, the *golden-mean shift*.

The fact that (up to scalar multiples) the Lucas sequence (L_n) is the only exactly realizable sequence satisfying the Fibonacci recurrence relation to some extent explains the familiar observation that (L_n) satisfies a great array of congruences.

Throughout, n will denote a positive integer and p, q distinct prime numbers.

2. EXISTENCE

An excellent introduction to the family of dynamical systems from which the example comes is the recent book by Lind and Marcus [4]. Let

$$X=\{\mathbf{x}=(x_k)\in\{0,1\}^{\mathbb{Z}}\,|\,x_k=1\Rightarrow x_{k+1}=0 \text{ for all } k\in\mathbb{Z}\}.$$

The set X is a compact metric space in a natural metric (see [4], Ch. 6, for the details). The set X may also be thought of as the set of all (infinitely long in both past and future) itineraries of a journey involving two locations (0 and 1), obeying the rule that from 1 you must travel to 0, and from 0 you must travel to either 0 or 1. Define the homeomorphism $f:X\to X$ to be the *left shift*,

$$(f(\mathbf{x}))_k=x_{k+1}\ \text{ for all } k\in\mathbb{Z}.$$

The dynamical system $f:X\to X$ is a simple example of a *subshift of finite type*. It is easy to check that the number of points of period n under this map is given by

$$\text{Per}_n(f)=\text{trace}(A^n), \tag{6}$$

where $A=\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$ (see [4], Prop. 2.2.12; the 0–1 entries in the matrix A correspond to the allowed transitions $0\to 0$ or 1; $1\to 0$ in the elements of X thought of as infinitely long journeys in a graph with vertices 0 and 1).

Lemma 2: If $b=3a$ in (4), then the corresponding sequence is exactly realizable.

Proof: A simple induction argument shows that (6) reduces to $\text{Per}_n(f)=L_n$ for $n\geq 1$, so the case $a=1$ is realized using the golden mean shift itself. For the general case, let $\overline{X}=X\times B$, where B is a set with a elements, and define $\bar{f}: \bar{X} \to \bar{X}$ by $\bar{f}(\mathbf{x}, y) = (f(\mathbf{x}), y)$. Then $\mathrm{Per}_n(\bar{f}) = a \times \mathrm{Per}_n(f)$, so we are done. □

The relation (3) must as a result hold for (L_n).

Corollary 3: $\sum_{d|n} \mu(n/d) L_d \equiv 0 \bmod n$ for all $n \geq 1$.

This has many consequences, a sample of which we list here. Many of these are, of course, well known (see [5], §2.IV) or follow easily from well-known congruences.

(a) Taking $n = p$ gives

$$L_p = F_{p-2} + 3F_{p-1} \equiv 1 \bmod p. \tag{7}$$

(b) It follows from (a) that

$$F_{p-1} \equiv 1 \bmod p \Leftrightarrow F_{p-2} \equiv -2 \bmod p, \tag{8}$$

which will be used below.

(c) Taking $n = p^k$ gives

$$L_{p^k} \equiv L_{p^{k-1}} \bmod p^k \tag{9}$$

for all primes p and $k \geq 1$.

(d) Taking $n = pq$ (a product of distinct primes) gives

$$L_{pq} + 1 \equiv L_p + L_q \bmod pq.$$

3. OBSTRUCTION

The negative part of Theorem 1 is proved as follows. Using some simple modular results on the Fibonacci numbers, we show that, if the sequence (U_n) defined by (4) is exactly realizable, then the property (3) forces the congruence $b \equiv 3a \bmod p$ to hold for infinitely many primes p, so (U_n) is a multiple of (L_n).

Lemma 4: For any prime p, $F_{p-1} \equiv 1 \bmod p$ if $p = 5m \pm 2$.

Proof: From Hardy and Wright (see [2], Theorem 180), we have that $F_{p+1} \equiv 0 \bmod p$ if $p = 5m \pm 2$. The identities $F_{p+1} = 2F_{p-1} + F_{p-2} \equiv 0 \bmod p$ and (7) imply that $F_{p-1} \equiv 1 \bmod p$. □

Assume now that the sequence (U_n) defined by (4) is exactly realizable. Applying (3) for n a prime p shows that $U_p - U_1 \equiv 0 \bmod p$, so by (5), $aF_{p-2} + bF_{p-1} \equiv a \bmod p$. If p is 2 or 3 mod 5, Lemma 4 implies that

$$(F_{p-2} - 1)a + b \equiv 0 \bmod p. \tag{10}$$

On the other hand, for such p, (8) implies that $F_{p-2} \equiv -2 \bmod p$, so (10) gives $b = 3a \bmod p$. By Dirichlet's theorem (or simpler arguments), there are infinitely many primes p with p equal to 2 or 3 mod 5, so $b = 3a \bmod p$ for arbitrarily large values of p. We deduce that $b = 3a$, as required.

4. REMARKS

(a) Notice that the example of the golden mean shift plays a vital role here. If it were not to hand, exhibiting a dynamical system with the required properties would require *proving* Corollary

3, and *a priori* we have no way of guessing or proving this congruence without using the dynamical system.

(b) The congruence (7) gives a different proof that $F_{p-1} \equiv 0$ or $1 \bmod p$ for $p \neq 2, 5$. If $F_{p-1} \equiv \alpha \bmod p$, then (7) shows that $F_{p-2} \equiv 1-3\alpha \bmod p$, so $F_p \equiv 1-2\alpha$. On the other hand, the recurrence relation gives the well-known equality

$$F_{p-2}F_p = F_{p-1}^2 + 1$$

(since p is odd), so $1-5\alpha+6\alpha^2 \equiv \alpha^2+1$, hence $5(\alpha^2-\alpha) \equiv 0 \bmod p$. Since $p \neq 5$, this requires that $\alpha^2 \equiv \alpha \bmod p$, so $\alpha \equiv 0$ or 1.

(c) The general picture of conditions on linear recurrence sequences that allow exact realization is not clear, but a simple first step in the Fibonacci spirit is the following question: For each $k \geq 1$, define a recurrence sequence $(U_n^{(k)})$ by

$$U_{n+k}^{(k)} = U_{n+k-1}^{(k)} + U_{n+k-2}^{(k)} + \cdots + U_n^{(k)}$$

with specified initial conditions $U_j^{(k)} = a_j$ for $1 \leq j \leq k$. The subshift of finite type associated to the 0–1 $k \times k$ matrix

$$A^{(k)} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ & & \ddots & & & \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 \end{bmatrix}$$

shows that the sequence $(U_n^{(k)})$ is exactly realizable if $a_j = 2^j - 1$ for $1 \leq j \leq k$. If the sequence is exactly realizable, does it follow that $a_j = C(2^j - 1)$ for $1 \leq j \leq k$ and some constant C? The special case $k = 1$ is trivial, and $k = 2$ is the argument above. Just as in Corollary 3, an infinite family of congruences follows for each of these multiple Fibonacci sequences from the existence of the exact realization.

(d) We are grateful to an anonymous referee for suggesting the following questions. Given a dynamical system $f : X \to X$ for which the quantities $\mathrm{Per}_n(f)$ are all finite, it is conventional to define the *dynamical zeta function*

$$\zeta_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Per}_n(f)\right),$$

which defines a complex function on the disc of radius

$$1 / \limsup_{n \to \infty} \mathrm{Per}_n(f)^{1/n}.$$

It is a remarkable fact that for many dynamical systems—indeed, all "hyperbolic" ones—the zeta function is a rational function. For example, the golden mean subshift of finite type used above has zeta function $\frac{1}{1-z-z^2}$. There are also sharp results that determine exactly what rational functions can arise as zeta functions of irreducible subshifts of finite type or of *finitely presented* systems—these are expansive quotients of subshifts of finite type. A simple application of Theorem 6.1 in [1], which describes the possible shape of zeta functions for finitely presented systems shows that the sequence a, $3a$, $4a$, $7a$, ... can be exactly realized by an irreducible subshift of finite type if and only if $a = 1$.

It is possible that the recent deep results of Kim, Ormes, and Roush [3] may eventually provide a complete description of linear recurrence sequences that are exactly realized by subshifts of finite type.

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