THE MULTIPLE SUM ON THE GENERALIZED LUCAS SEQUENCES

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The computation of the multiple sum on a linear recurrence sequence is an interesting question. Many fine results have been given. This paper will establish a computational formula for the multiple sum on the generalized Lucas sequence. A new method will be used and some congruence relations will be given.

We define a linear recurrence sequence $W_n = W_n(a, b; p, q)$, $n = 0, 1, \dots$, as

$$W_n = pW_{n-1} - qW_{n-2} \quad (n \geq 2)$$
$$W_0 = a, \; W_1 = b.$$

We consider the sequence

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q). \end{cases}$$

Then U_n and V_n are called the generalized Fibonacci sequence and the generalized Lucas sequence. Their Binet formulas are, respectively,

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}.$$

In [2], W. Zhang gave a computational formula involving the multiple sum on the generalized Fibonacci sequence when $U_0 = 0$.

In this paper, we shall use another method (formal power) to establish a computational formula for the multiple sum on the generalized Lucas sequence, i.e.,

$$\sum_{a_1 + a_2 + \cdots + a_k = n} V_{a_1} V_{a_2} \cdots V_{a_k},$$

where the summation is taken over all n-tuples with positive coordinates $(a_1, a_2, \dots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$.

The generating function of the Generalized Lucas sequence $\{V_n\}_0^\infty$ is

$$H(x) = \sum_{n \geq 0} V_n x^n = \frac{2 - px}{1 - px + qx^2}.$$

Let

$$H_k(x) = \left(\frac{2 - px}{1 - px + qx^2}\right)^k = \sum_{n \geq 0} V_n^{(k)} x^n.$$

Obviously, $V_n^{(1)} = V_n$. Then

$$\sum_{a_1 + a_2 + \cdots + a_m = n} V_{a_1}^{(k_1)} V_{a_2}^{(k_2)} \cdots V_{a_m}^{(k_m)} = V_n^{(k_1 + k_2 + \cdots + k_m)}.$$

If we take $k_1 = k_2 = \cdots = k_m = 1$, we obtain the following lemma.

Lemma: $\displaystyle\sum_{a_1+a_2+\cdots+a_m=n} V_{a_1}V_{a_2}\cdots V_{a_m} = V_n^{(m)}.$

Theorem 1: Let $V_n^{(k)}$ be defined as above. Then

$$V_n^{(k+1)} = \frac{1}{k(p^2-4q)}\{4(n+2)V_{n+2}^{(k)} - 2p(2n+k+2)V_{n+1}^{(k)} + p^2(n+k)V_n^{(k)}\}.$$

Proof: We note the following equalities:

$$\begin{aligned}\frac{d}{dx}(H_k(x)) &= \frac{d}{dx}\left(\frac{2-px}{1-px+qx^2}\right)^k\\ &= k\left(\frac{2-px}{1-px+qx^2}\right)^{k-1}\frac{-p(1-px+qx^2)+(2-px)(p-2qx)}{(1-px+qx^2)^2}\\ &= k\left(\frac{2-px}{1-px+qx^2}\right)^{k-1}\frac{-p+p^2x-pqx^2+2p-4qx-p^2x+2pqx^2}{(1-px+qx^2)^2}\\ &= k\left(\frac{2-px}{1-px+qx^2}\right)^{k-1}\frac{p-4qx+pqx^2}{(1-px+qx^2)^2}\\ &= k\left(\frac{2-px}{1-px+qx^2}\right)^{k-1}\frac{p(1-px+qx^2)+(p^2-4q)x}{(1-px+qx^2)^2}\\ &= \frac{kp}{2-px}\left(\frac{2-px}{1-px+qx^2}\right)^k + \frac{k(p^2-4q)x}{(2-px)^2}\left(\frac{2-px}{1-px+qx^2}\right)^{k+1}.\end{aligned}$$

Thus,

$$kx(p^2-4q)\left(\frac{2-px}{1-px+qx^2}\right)^{k+1} = (2-px)^2\frac{d}{dx}\left(\frac{2-px}{1-px+qx^2}\right)^k - kp(2-px)\left(\frac{2-px}{1-px+qx^2}\right)^k.$$

So

$$kx(p^2-4q)\sum_{n\geq 0}V_n^{(k+1)}x^n = (4-4px+p^2x^2)\sum_{n\geq 0}nV_n^{(k)}x^{n-1} - kp(2-px)\sum_{n\geq 0}V_n^{(k)}x^n.$$

Comparing coefficients on both sides of the equation, we have

$$\begin{aligned}k(p^2-4q)V_{n-1}^{(k+1)} &= 4(n+1)V_{n+1}^{(k)} - 4npV_n^{(k)} + p^2(n-1)V_{n-1}^{(k)} - 2kpV_n^{(k)} + kp^2V_{n-1}^{(k)}\\ &= 4(n+1)V_{n+1}^{(k)} - 2p(2n+k)V_n^{(k)} + p^2(n+k-1)V_{n-1}^{(k)}.\end{aligned}$$

This completes the proof. □

Taking $k = 1, 2, 3, 4$ in the lemma and using Theorem 1, we obtain the following results.

Theorem 2: Let (V_n) be defined as above. We have the following identities:

(a) $\displaystyle\sum_{a+b=n} V_aV_b = \frac{1}{p^2-4q}\{2pV_{n+1} + [(n+1)(p^2-4q)-4q]V_n\}.$

(b) $\sum_{a+b+c=n} V_aV_bV_c = \frac{n+2}{2(p^2-4q)}\{6pV_{n+1}+[(n+1)(p^2-4q)-12q]V_n\}.$

(c) $\sum_{a+b+c+d=n} V_aV_bV_cV_d = \frac{1}{3!(p^2-4q)}\{12[(n+3)_2(p^2-4q)+(n+1)(p^2-4q)-2q]V_{n+2}$
$+[(n+3)_3(p^2-4q)^2-12q(n+3)_2(p^2-4q)+24q^2]V_n\}.$

(d) $\sum_{a+b+c+d+e=n} V_aV_bV_cV_dV_e = \frac{1}{4!(p^2-4q)^2}\{48(n+1)(n+4)V_{n+4}-4[12q(n+2)(n^2+10n+23)$
$-2(n+4)_3(p^2-4q)-3p^2(n+4)(n^2+6n+7)+6nq]V_{n+2}$
$+(n+4)[48q^2(n+3)^2+(n+3)_3(p^2-4q)^2-8q(n+3)_2(p^2-4q)$
$-12p^2q(n+3)_2+24q^2]V_n\}.$

Here, $(n)_k = n(n-1)(n-2)\cdots(n-k+1)$.

Theorem 3: Under the conditions of Theorem 2, we have the following:

(a) $2V_{n+2}-2qV_n \equiv 0 \pmod{p^2-4q}$.

(b) $12[(n+3)_2(p^2-4q)+(n+1)(p^2-4q)-2q]V_{n+2}$
$+[(n+3)_3(p^2-4q)^2-12q(n+3)_2(p^2-4q)+24q^2]V_n \equiv 0 \pmod{3!(p^2-4q)^2}$.

(c) $(n+4)[48q^2(n+3)^2+(n+3)_3(p^2-4q)^2-8q(n+3)_2(p^2-4q)-12p^2q(n+3)_2+24q^2]V_n$
$-4[12q(n+2)(n^2+10n+23)-2(n+4)_3(p^2-4q)-3p^2(n+4)(n^2+6n+7)+6nq]V_{n+2}$
$+48(n+1)(n+4)V_{n+4} \equiv 0 \pmod{4!(p^2-4q)^2}$.

Proof: Use Theorem 2(a), (c), (d).

Taking $p=-q=1$, $V_n = L_n$ is the Lucas sequence, i.e., $L_0=2$, $L_1=1$, $L_2=3$, $L_3=4$, ... Thus, from Theorem 2, we obtain Corollaries 1 and 2.

Corollary 1: Let (L_n) be the Lucas sequence. Then we have the following:

(a) $\sum_{a+b=n} L_aL_b = \frac{1}{5}\{2L_{n+1}+(5n+9)L_n\}.$

(b) $\sum_{a+b+c=n} L_aL_bL_c = \frac{n+2}{10}\{6L_{n+1}+(5n+17)L_n\}.$

(c) $\sum_{a+b+c+d=n} L_aL_bL_cL_d = \frac{1}{150}\{12(5n^2+30n+37)L_{n+1}+(25n^3+270n^2+935n+978)L_n\}.$

(d) $\sum_{a+b+c+d+e=n} L_aL_bL_cL_dL_e = \frac{1}{600}\{48(n+1)(n+4)L_{n+4}+4(25n^3+264n^2+875n+876)L_{n+2}\}$
$+\frac{1}{600}[(n+3)(n+4)(25n^2+175n+298)+24(n+4)]L_n.$

Corollary 2: Let (L_n) be the Lucas sequence. Then we have the following congruences:

(a) $L_{n+2}+L_n \equiv 0 \pmod 5$.

(b) $12(5n^2+30n+37)L_{n+1}+(25n^3+120n^2+35n+78)L_n \equiv 0 \pmod{150}$.

(c) $48(n+1)(n+4)L_{n+4}+4(25n^3+264n^2+875n+876)L_{n+2}$
$+[(n+3)(n+4)(25n^2+175n+298)+24(n+4)]L_n \equiv 0 \pmod{600}$.

First, we gave the multiple sum on the generalized Lucas sequence. Then, we discussed the multiple sum on the even generalized Lucas sequence. Now

$$\sum_{n\geq 0} V_{2n}x^{2n} = \frac{1}{2}\left\{\frac{2-px}{1-px+qx^2}+\frac{2+px}{1+px+qx^2}\right\} = \frac{2-(p^2-2q)x^2}{1-(p^2-2q)x^2+q^2x^4}.$$

We use methods similar to those employed above. Let

$$\sum_{n\geq 0} R_{2n}^{(k)}x^n = \left(\frac{2-(p^2-2q)x}{1-(p^2-2q)x+q^2x^2}\right)^k.$$

Obviously, $R_{2n}^{(1)} = V_{2n}$,

$$\sum_{a_1+a_2+\cdots+a_m=n} V_{2a_1}V_{2a_2}\cdots V_{2a_m} = R_{2n}^{(m)},$$

$$R_{2n}^{(k+1)} = \frac{1}{kp^2(p^2-4q)}\{4(n+2)R_{2n+4}^{(k)} - 2(p^2-2q)(2n+k+2)R_{2n+2}^{(k)} + (p^2-2q)^2(n+k)R_{2n}^{(k)}\}.$$

Hence, we have the following theorems.

Theorem 4: $\displaystyle\sum_{a+b=n} V_{2a}V_{2b} = \frac{1}{p^2(p^2-4q)}\{2(p^2-2q)V_{2n+2}+[(n+1)p^2(p^2-4q)-4q^2]V_{2n}\}$,

$$\sum_{a+b+c=n} V_{2a}V_{2b}V_{2c} = \frac{1}{2p^2(p^2-4q)}\{6(p^2-2q)V_{2n+2}+[(n+1)p^2(p^2-4q)-12q^2]V_{2n}\}.$$

Theorem 5: $2(p^2-2q)V_{2n+2}-4q^2V_{2n} \equiv 0 \pmod{p^2(p^2-4q)}$.

$$6(p^2-2q)V_{2n+2}+\{p^2(n+1)(p^2-4q)-12q^2]V_{2n} \equiv 0 \pmod{2p^2(p^2-4q)}.$$

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