MORGAN-VOYCE CONVOLUTIONS

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1. GENERATING FUNCTIONS

Morgan-Voyce Generators

Much has been written lately about the four Morgan-Voyce polynomials $B_n(x)$, $b_n(x)$, $C_n(x)$, and $c_n(x)$. Basic properties of these polynomials are developed in [2], which contains appropriate reference material.

The main purpose of this paper is to investigate the simplest features of the convolutions of the Morgan-Voyce polynomials and their corresponding numbers occurring when $x=1$. Our Morgan-Voyce polynomials are defined [2] in terms of generating functions thus:

$$\sum_{n=1}^{\infty} B_n(x)y^{n-1} = [1-\overline{(2+x)y-y^2}]^{-1} = g,\ B_0(x)=0, \tag{1.1}$$

$$\sum_{n=0}^{\infty} C_n(x)y^n = [2-(2+x)y]g, \tag{1.2}$$

$$\sum_{n=1}^{\infty} b_{n-1}(x)y^{n-1} = [1-(1+x)y]g, \tag{1.3}$$

$$\sum_{n=0}^{\infty} c_n(x)y^n = [-1+(3+x)y]g, \tag{1.4}$$

where, in (1.1)-(1.4), the functional notation $g(x,y)\equiv g$ has been dropped in the interest of simplicity. So, g (1.1) may be said to be the "single parent" progenitor of the family (1.2)-(1.4)!

Partial differentiation with respect to x (Section 5), which is a second feature of this paper, provides us with deeper insights into the essential nature of the polynomials. Two related papers could indeed have evolved from this paper but it is thought more desirable to preserve unity and cohesiveness.

Motivation

Stimuli for pursuing this investigation are:

(i) in mountaineering language, "it [the challenge] is there!" and
(ii) it increases our knowledge of convolution analysis beyond that already established for other well-known polynomials.

Initial Conditions

All the convolution number sequences displayed in (2.2a), (2.3a); (3.2a), (3.3a); (4.2a), (4.3a); (5.2a), (5.3a) for $B_n^{(k)}(x), C_n^{(k)}(x), b_n^{(k)}(x), c_n^{(k)}(x)$, respectively, when $k=1,2$, have been checked against those obtainable from the general formulas in [1] which were determined by means of Cauchy products. This signifies that our generating function definitions must, when $x=1$, produce exactly the same two initial numbers of each sequence as are specified in [1].

2. CONVOLUTIONS FOR $B_n(x)$

Definitions

The k^{th} *convolution polynomials* $B_n^{(k)}(x)$ of $B_n(x)$ are defined by

$$\sum_{n=1}^{\infty} B_n^{(k)}(x)y^{n-1} = g^{k+1}, \; B_0^{(k)}(x) = 0, \tag{2.1}$$

$$= \left(\sum_{n=1}^{\infty} B_n(x)y^{n-1}\right)^{k+1}, \tag{2.1a}$$

so that $B_n^{(0)}(x) \equiv B_n(x)$.

Correspondingly, the k^{th} *convolution numbers* $B_n^{(k)}(1) \equiv B_n^{(k)}$ arise in the special case when $x = 1$.

Examples

$k = 1$

$$\begin{aligned} &B_1^{(1)}(x) = 1, \; B_2^{(1)}(x) = 4+2x, \; B_3^{(1)}(x) = 10+12x+3x^2, \\ &B_4^{(1)}(x) = 20+42x+24x^2+4x^3, \; B_5^{(1)}(x) = 35+112x+108x^2+40x^3+5x^4, \ldots. \end{aligned} \tag{2.2}$$

$k = 2$

$$\begin{aligned} &B_1^{(2)}(x) = 1, \; B_2^{(2)}(x) = 6+3x, \; B_3^{(2)}(x) = 21+24x+6x^2, \\ &B_4^{(2)}(x) = 56+108x+60x^2+10x^3, \; B_5^{(2)}(x) = 126+360x+330x^2+120x^3+15x^4, \ldots. \end{aligned} \tag{2.3}$$

Special Cases

$$\{B_n^{(1)}\}_1^{\infty} = 1, 6, 25, 90, 300, \ldots. \tag{2.2a}$$

$$\{B_n^{(2)}\}_1^{\infty} = 1, 9, 51, 234, 961, \ldots. \tag{2.3a}$$

Larger values of k and n clearly involve cumbersome expressions which do not excite our interest.

Recurrence Relations

Immediately from (1.1) and (2.1) we deduce that

$$B_n^{(k)}(x) = B_n^{(k+1)}(x) - (2+x)B_{n-1}^{(k+1)}(x) + B_{n-2}^{(k+1)}(x) \tag{2.4}$$

with the simplest instance ($k = 0$) being

$$B_n(x) = B_n^{(1)}(x) - (2+x)B_{n-1}^{(1)}(x) + B_{n-2}^{(1)}(x). \tag{2.4a}$$

Partial differentiation with respect to y in (2.1) and comparison of coefficients of y^{n-2} leads to

$$(n-1)B_n^{(k)}(x) = (k+1)\{(2+x)B_{n-1}^{(k+1)}(x) - 2B_{n-2}^{(k+1)}(x)\}. \tag{2.5}$$

Amalgamating (2.4) and (2.5) and replacing k by $k-1$, we obtain the reduction

$$(n-1)B_n^{(k)}(x) = (n+k-1)(2+x)B_{n-1}^{(k)}(x) - (n+2k-1)B_{n-2}^{(k)}(x). \tag{2.6}$$

Recurrence (2.6) enables us to consolidate a table for $B_n^{(k)}(x)$, given $x = 1$, from two previously known successive values. Substitution of $k = 0$ reduces (2.6) to the defining recurrence for $B_n(x)$. Furthermore, $k = 0$ in (2.5) produces the simple link ($n \to n+1$)

$$nB_{n+1}(x) = (2+x)B_n^{(1)}(x) - 2B_{n-1}^{(1)}(x). \tag{2.5a}$$

Further partial differentiation, but this time with respect to x, will be investigated for all the Morgan-Voyce polynomials separately in Section 5.

3. CONVOLUTIONS FOR $C_n(x)$

Coming now to $C_n(x)$ we find ourselves enmeshed in more complicated algebra than that for $B_n(x)$, by virtue of the definition (1.2).

Definitions

The k^{th} *convolution polynomials* $C_n^{(k)}(x)$ of $C_n(x)$ are defined by

$$\sum_{n=0}^{\infty} C_n^{(k)}(x)y^n = [2-(2+x)y]^{k+1}g^{k+1} \tag{3.1}$$

$$= \left(\sum_{n=0}^{\infty} C_n(x)y^n\right)^{k+1}, \tag{3.1a}$$

so that $C_n^{(0)}(x) \equiv C_n(x)$.

Correspondingly, the k^{th} *convolution numbers* $C_n^{(k)}(1) \equiv C_n^{(k)}$ arise when $x = 1$.

Examples

$\boldsymbol{k = 1}$

$$\begin{aligned} &C_0^{(1)}(x) = 4,\ C_1^{(1)}(x) = 4(2+x),\ C_2^{(1)}(x) = 12+20x+5x^2, \\ &C_3^{(1)}(x) = 16+56x+36x^2+6x^3,\ C_4^{(1)}(x) = 20+120x+142x^2+56x^3+7x^4, \ldots \end{aligned} \tag{3.2}$$

$\boldsymbol{k = 2}$

$$\begin{aligned} &C_0^{(2)}(x) = 8,\ C_1^{(2)}(x) = 12(2+x),\ C_2^{(2)}(x) = 48+72x+18x^2, \\ &C_3^{(2)}(x) = 80+240x+150x^2+25x^3,\ C_4^{(2)}(x) = 120+600x+678x^2+264x^3+33x^4, \ldots \end{aligned} \tag{3.3}$$

Special Cases

$$\{C_n^{(1)}\}_0^{\infty} = 4, 12, 37, 114, 345, \ldots \tag{3.2a}$$

$$\{C_n^{(2)}\}_0^{\infty} = 8, 36, 138, 495, 1695, \ldots \tag{3.3a}$$

Recurrence Relations

Taken together, (2.1) and (3.1) give rise, when $k = 1$, to

$$C_{n-1}^{(1)}(x) = 4B_n^{(1)}(x) - 4(2+x)B_{n-1}^{(1)}(x) + (2+x)^2 B_{n-2}^{(1)}(x). \tag{3.4}$$

Differentiate (1.2) partially with respect to y and equate coefficients of y^{n-1}. After simplification, the algebra reduces to

$$nC_n(x) = (2+x)B_n^{(1)}(x) - 4B_{n-1}^{(1)}(x) + (2+x)B_{n-2}^{(1)}(x). \tag{3.5}$$

Uniting (3.4) and (3.5), we establish, on tidying up, that

$$n(2+x)C_n(x) = (4+x)xB_n^{(1)}(x) + C_{n-1}^{(1)}(x). \tag{3.6}$$

Multiply numerator and denominator of (3.1), when $k = 0$, by g (1.1). Simplification then shows, by (2.1), that

$$C_{n-1}(x) = 2B_n^{(1)}(x) - 3(2+x)B_{n-1}^{(1)}(x) + (6+4x+x^2)B_{n-2}^{(1)}(x) - (2+x)B_{n-3}^{(1)}(x). \tag{3.7}$$

Extending (3.5) to $k=2$, we quickly get

$$C_{n-1}^{(2)}(x)=8B_n^{(2)}(x)-12(2+x)B_{n-1}^{(2)}+6(2+x)^2B_{n-2}^{(2)}(x)-(2+x)^3B_{n-3}^{(2)}(x). \tag{3.8}$$

Beyond this, the formulas become even less algebraically attractive. Enchantment and time are lacking to pursue this unproductive activity.

4. CONVOLUTIONS FOR $b_n(x)$

Definitions

The k^{th} *convolution polynomials* $b_n^{(k)}(x)$ of $b_n(x)$ are defined by

$$\sum_{n=1}^{\infty} b_{n-1}^{(k)}(x)y^{n-1}=\{1-(1+x)y\}^{k+1}g^{k+1} \quad (\text{so } b_0^{(k)}(x)=1) \tag{4.1}$$

$$=\left(\sum_{n=1}^{\infty} b_{n-1}(x)y^{n-1}\right)^{k+1}. \tag{4.1a}$$

In particular, when $x=1$, the k^{th} *convolution numbers* $b_n^{(k)}(1)\equiv b_n^{(k)}$ emerge.

Examples

$k=1$

$$\begin{aligned}&b_1^{(1)}(x)=2,\ b_2^{(1)}(x)=3+2x,\ b_3^{(1)}(x)=4+8x+2x^2,\\ &b_4^{(1)}(x)=5+20x+13x^2+2x^3,\dots\end{aligned} \tag{4.2}$$

$k=2$

$$\begin{aligned}&b_1^{(2)}(x)=3,\ b_2^{(2)}(x)=6+3x,\ b_3^{(2)}(x)=10+15x+3x^2,\\ &b_4^{(2)}(x)=15+45x+24x^2+3x^3,\dots\end{aligned} \tag{4.3}$$

Special Cases

$$\{b_n^{(1)}\}_0^{\infty}=1,2,5,14,40,\dots \tag{4.2a}$$

$$\{b_n^{(2)}\}_0^{\infty}=1,3,9,28,87,\dots \tag{4.3a}$$

Recurrence Relations

Put $k=1$ in (4.1). Then we immediately construct the recurrence

$$b_n^{(1)}(x)=B_{n+1}^{(1)}(x)-2(1+x)B_n^{(1)}(x)+(1+x)^2B_{n-1}^{(1)}(x). \tag{4.4}$$

Partially differentiate (4.1) with respect to y. Then

$$nb_n(x)=B_n^{(1)}(x)-2B_{n-1}^{(1)}(x)+(1+x)B_{n-2}^{(1)}(x). \tag{4.5}$$

Together, with suitable adjustment, (4.4) and (4.5) produce

$$nb_n(x)=b_{n-1}^{(1)}(x)+2xB_{n-1}^{(1)}(x)-(x+x^2)B_{n-2}^{(1)}(x). \tag{4.6}$$

Next, let us multiply numerator and denominator of (4.1), when $k=1$, by g (1.1). Upon the requisite algebraic manipulation with application of $b_n^{(2)}(x)$ given by (4.1), when $k=2$, namely,

$$b_n^{(2)}(x)=B_{n+1}^{(2)}(x)-3(1+x)B_n^{(2)}(x)+3(1+x)^2B_{n-1}^{(2)}(x)-(1+x)^3B_{n-2}^{(2)}(x), \tag{4.7}$$

it transpires that

$$b_n^{(2)}(x)=b_n^{(1)}(x)+B_n^{(2)}(x)-(3+2x)B_{n-1}^{(2)}(x)+(3+4x+x^2)B_{n-2}^{(2)}(x)-(1+x)^2B_{n-3}^{(2)}(x). \tag{4.8}$$

Caveat! Anticipating (5.1) we might have been tempted to use the formula $b_n(x) = B_n(x) - B_{n-1}(x)$ [2, (2.13) : $x = 1$] to derive the valid generating function $\sum_{n=1}^{\infty} b_n(x)y^{n-1} = (1-y)g$. However, the difficulty here for convolutions is that the first element defined is $b_1(x) = 1$. What we need is $b_0(x) = 1$ to be covered by the definition. Consequently, we must abide by (4.1).

5. CONVOLUTIONS FOR $c_n(x)$

Definitions

Care must be taken when we come to deal with the convolutions of the last of our four Morgan-Voyce polynomials. Our problem with $c_n(x)$ as defined in (1.4) is that $c_0(x) = -1$. But we do not want negative numbers as part of convolutions. So we begin the sequence for $c_n(x)$ with $c_1(x) = 1$.

Recalling [2, (3.7)] that $c_n(x) = B_n(x) + B_{n-1}(x)$, we define the k^{th} *convolution polynomials* $c_n^{(k)}(x)$ of $c_n(x)$ to be given by ($n \geq 1$)

$$\sum_{n=1}^{\infty} c_n^{(k)}(x)y^{n-1} = (1+y)^{k+1}g^{k+1}. \tag{5.1}$$

Substitution of $x = 1$ engenders the k^{th} *convolution numbers* $c_n^{(k)}(1) \equiv c_n^{(k)}$.

Examples

$\boldsymbol{k = 1}$

$$\begin{aligned} &c_1^{(1)}(x) = 1,\ c_2^{(1)}(x) = 6+2x,\ c_3^{(1)}(x) = 19+16x+3x^2, \\ &c_4^{(1)} = 44+68x+30x^2+4x^3,\ c_5^{(1)} = 85+208x+159x^2+48x^3+5x^4, \dots . \end{aligned} \tag{5.2}$$

$\boldsymbol{k = 2}$

$$\begin{aligned} &c_1^{(2)}(x) = 1,\ c_2^{(2)}(x) = 9+3x,\ c_3^{(2)}(x) = 42+33x+6x^2, \\ &c_4^{(2)}(x) = 138+189x+78x^2+10x^3,\ c_5^{(2)}(x) = 363+759x+528x^2+150x^3+15x^4, \dots . \end{aligned} \tag{5.3}$$

Special Cases

$$\{c_n^{(1)}\}_1^{\infty} = 1, 8, 38, 146, 505, \dots . \tag{5.2a}$$

$$\{c_n^{(2)}\}_1^{\infty} = 1, 12, 81, 415, 1815, \dots . \tag{5.3a}$$

Recurrence Relations

From (5.1) and (1.1) we have automatically

$$c_n^{(1)}(x) = B_n^{(1)}(x) + 2B_{n-1}^{(1)}(x) + B_{n-2}^{(1)}(x). \tag{5.4}$$

Partial differentiation in (5.1) with respect to y, in conjunction with (1.1), and $n \to n+1$, produces

$$nc_{n+1}(x) = (3+x)B_n^{(1)}(x) - 2B_{n-1}^{(1)}(x) - B_{n-2}^{(1)}(x). \tag{5.5}$$

Joining (5.4) and (5.5) ensures the neat nexus

$$c_n^{(1)}(x) = (4+x)B_n^{(1)}(x) - nc_{n+1}(x). \tag{5.6}$$

Next, taking $k = 0$, multiply numerator and denominator in (5.1) by g. Organizing the resulting material and applying (1.1) then establishes the result:

$$c_{n+1}(x) = B_{n+1}^{(1)}(x) - (1+x)[B_n^{(1)}(x) + B_{n-1}^{(1)}(x)] + B_{n-2}^{(1)}(x). \tag{5.7}$$

6. PARTIAL DIFFERENTIATION

In this section, partial differentiation is performed **only with respect to x.**

Notation

Successive orders of partial differentiation (first, second, third, ..., k^{th}) will be represented by superscript primes $'$, $''$, $'''$, ..., k primes, where the unbracketed superscript k is to be clearly distinguished from the bracketed k^{th} convolution order symbol superscript (k). Thus, we will have

$$B_n'(x)=\frac{\partial B_n(x)}{\partial x},\ B_n''(x)=\frac{\partial^2 B_n(x)}{\partial x^2},\ldots,B_n^k(x)=\frac{\partial^k B_n(x)}{\partial x^k}.$$

Likewise for $C_n(x)$, $b_n(x)$, and $c_n(x)$.

I. $B_n^k(x)$: Equate appropriate coefficients using (1.1) in

$$\sum_{n=1}^{\infty}B_n'(x)y^{n-1}=yg^2=\sum_{m=0}^{\infty}B_m^{(1)}(x)y^m$$

unfolding the nice result

$$B_n'(x)=B_{n-1}^{(1)}(x). \tag{6.1}$$

Repetition of the process gives

$$B_n''(x)=2B_{n-2}^{(2)}(x). \tag{6.1a}$$

Generally,

$$B_n^k(x)=k!\,B_{n-k}^{(k)}(x). \tag{6.1b}$$

Temporarily revert to $B_n^{(2)}(x)$. Then we may write

$$\sum_{n=1}^{\infty}B_n^{(2)}(x)y^{n-1}=[\{1-(2+x)y+y^2\}+\{(2+x)y-1\}]g^3=\sum_{n=1}^{\infty}B_n^{(1)}(x)y^{n-1}+\{(2+x)y-1\}g^3,$$

whence

$$B_n^{(2)}(x)=B_n^{(1)}(x)+(2+x)B_{n-1}^{(2)}(x)-B_{n-2}^{(2)}(x). \tag{6.2}$$

Accordingly, (6.1a) and (6.2) conjoined give

$$B_n^{(1)}(x)=B_n^{(2)}(x)-(2+x)B_{n-1}^{(2)}(x)+B_{n-2}^{(2)}(x) \tag{6.3}$$

which is (2.4) when $k=1$.

Two pleasant theorems now conclude this subsection.

Theorem 1: $B_{n+2}''(x)-B_n''(x)=(n+1)B_n^{(1)}(x)$.

Proof:

$$\begin{aligned}B_{n+2}''(x)-B_n''(x)&=2B_n^{(2)}(x)-2B_{n-2}^{(2)}(x)\quad\text{by (2.12)}\\&=2B_n^{(1)}(x)+2\{(2+x)B_{n-1}^{(2)}(x)-B_{n-2}^{(2)}(x)\}-2B_{n-2}^{(2)}(x)\quad\text{by (2.7)}\\&=2B_n^{(1)}(x)+2\{(2+x)B_{n-1}^{(2)}(x)-2B_{n-2}^{(2)}(x)\}\\&=2B_n^{(1)}(x)+(n-1)B_n^{(1)}(x)\quad\text{by (2.8)}\\&=(n+1)B_n^{(1)}(x).\end{aligned}$$

Corollary 1: $\sum_{n=2}^{m}nB_{n-1}^{(1)}(x)=B_m''(x)+B_{m+1}''(x)$.

More generally,

Theorem 2: $B_{n+2}^{k}(x)-B_{n}^{k}(x)=(k-1)!(n+1)B_{n+2-k}^{(k-1)}(x)$.

In particular, we have $B'_{n+2}(x)-B'_{n}(x)=(n+1)B_{n+1}(x)\ (=C'_{n+1}(x))$ as we expect from [2, (3.9), (3.24)].

II. $C_n^k(x)$**:** Consider the relation $C_n(x)=B_{n+1}(x)-B_{n-1}(x)$ [2, (3.9)]. Redefine (1.2) in this context to assert

$$\sum_{n=1}^{\infty} C_n(x)y^{n+1}=(y-y^3)g-y,\ C_0(x)=2. \tag{6.4}$$

Elementary processes then, with [2, (3.24)] produce

$$C'_n(x)=B_n^{(1)}(x)-B_{n-2}^{(1)}(x)=nB_n(x), \tag{6.5}$$

$$C''_n(x)=2(B_{n-1}^{(2)}(x)-B_{n-3}^{(2)}(x))=nB_{n-1}^{(1)}(x)=nB'_n(x), \tag{6.6}$$

culminating in

$$C_n^k(x)=k!(B_{n-k+1}^{(k)}(x)-B_{n-k-1}^{(k)}(x))=n(k-1)!B_{n-k+1}^{(k-1)}(x), \tag{6.7}$$

whence

$$\sum_{n=k+1}^{m} C_n^k(x)=k!(B_{m-k+1}^{(k)}(x)+B_{m-k}^{(k)}(x)-1). \tag{6.8}$$

In particular ($k=1$),

$$\sum_{n=1}^{m} C'_n=B_m^{(1)}(x)+B_{m-1}^{(1)}(x). \tag{6.8a}$$

Analogously to Theorem 2 there is

$$C_{n+2}^k(x)-C_n^k(x)=nC_{n+1}^{k-1}(x)+2B_{n+2}^{k-1}(x), \tag{6.9}$$

which can be expressed in convolution form. Proof of the assertion (6.9) is left to the reader.

III. $b_n^k(x)$**:** Convolutions of $b_n(x)$ do not appear in this section (see the *Caveat* in Section 4), so we may, on making use of [2, (2.13)], choose the definition

$$\sum_{n=1}^{\infty} b_n(x)y^{n-1}=(1-y)g,\quad b_0(x)=1. \tag{6.10}$$

Then, by (1.1),

$$b'_n(x)=B_{n-1}^{(1)}(x)-B_{n-2}^{(1)}(x)=B'_n(x)-B'_{n-1}(x), \tag{6.11}$$

$$b''_n(x)=2(B_{n-2}^{(2)}(x)-B_{n-3}^{(2)}(x))=B''_n(x)-B''_{n-1}(x). \tag{6.12}$$

Eventually, and generally,

$$b_n^k(x)=k!(B_{n-k}^{(k)}(x)-B_{n-k-1}^{(k)}(x))=B_n^k(x)-B_{n-1}^k(x). \tag{6.13}$$

Summation discloses that

$$\sum_{n=1}^{m} b'_n(x)=B_{m-1}^{(1)}(x)=B'_m(x) \tag{6.14}$$

while

$$\sum_{n=k+1}^{m} b_n^k(x)=k!B_{m-k}^{(k)}(x)=B_m^k(x). \tag{6.15}$$

From $nB_n(x) = C_n'(x) = b_n'(x) + b_{n+1}'(x)$, we may deduce after a little rearrangement that

$$\sum_{n=1}^{m}(-1)^{m+n}nB_n(x) = b_{m+1}'(x) \tag{6.16}$$

which can be generalized to $b_{m+1}^k(x)$.

IV. $c_n^k(x)$: Appealing to [2, (3.7)], we take

$$\sum_{n=0}^{\infty}c_n(x)y^n = (1+y)g. \tag{6.17}$$

Following the procedure in III, we rapidly reach the general situation:

$$c_n^k(x) = k!(B_{n-k}^{(k)}(x) + B_{n-k-1}^{(k)}(x)) = B_n^k(x) + B_{n-1}^k(x). \tag{6.18}$$

From $nB_n^k(x) = C_n^{k+1}(x) = c_{n+1}^{k+1}(x) - c_n^{k+1}(x)$ (see [2, (3.11)]), it then transpires that

$$\sum_{n=1}^{m}nB_n^k(k) = c_{m+1}^{k+1}(x). \tag{6.19}$$

Suppose $k = 1$ in (6.18). Addition then reveals that

$$\sum_{n=2}^{m}(-1)^n c_n'(x) = (-1)^m B_{m-1}^{(1)}(x) = (-1)^m B_m'(x), \tag{6.20}$$

whence, by (6.16),

$$\sum_{n=2}^{m}(-1)^n c_n'(x) = (-1)^m \sum_{n=2}^{m} b_n'(x). \tag{6.21}$$

7. CONCLUSION

Undertaking a thorough investigation of the latent features of the mixed foursome of Morgan-Voyce polynomials is a task of rather Herculean proportions, but no doubt somewhat more satisfying than cleansing the Augean stables. One challenge confronting us is an examination of the rising and falling diagonal polynomials associated with the Morgan-Voyce polynomials. For a related study of this kind of project, the recent paper [3], containing many references, is strongly suggested.

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REFERENCES

1. V. E. Hoggatt, Jr., & G. E. Bergum. "Generalized Convolution Arrays." *The Fibonacci Quarterly* **13.3** (1975):193-98.
2. A. F. Horadam. "New Aspects of Morgan-Voyce Polynomials." *Applications of Fibonacci Numbers* **7:**161-76. Ed. G. E. Bergum, et al. Dordrecht: Kluwer, 1998.
3. M. N. S. Swamy. "Generalized Fibonacci and Lucas Polynomials and Their Associated Diagonal Polynomials." *The Fibonacci Quarterly* **37.3** (1999):213-22.

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