PASCAL DECOMPOSITIONS OF ARITHMETIC AND CONVOLUTION ARRAYS IN MATRICES

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1. INTRODUCTION

Bicknell and Hoggatt [1]-[6] , [9] published several articles in the 1970s involving matrices made up of generalized arithmetic progressions and convolutions of sequences with first term one. We give a new proof of their result using a novel decomposition of such matrices and then extend their result to convolution matrices of sequences whose first term does not equal one. In the process, we gain an increased understanding of the underlying structures of such matrices. We also note that these results should be readily extensible to a class of matrices recently discussed by Ollerton and Shannon [11].

2. ARITHMETIC PROGRESSION MATRICES

In [5] and others, Bicknell and Hoggatt define an *arithmetic progression of* r^{th} *order*, or $(AP)_r$, as any sequence of numbers whose r^{th} row of differences is a nonzero constant while the $(r-1)^{\text{st}}$ is not. The constant number in the r^{th} row is called the *constant of the progression*. The sequence itself is the zeroth row of differences, so a constant nonzero sequence is an $(AP)_0$. They then give the following theorem.

Theorem 1 ("Eves' Theorem"): Let A be an $n\times n$ matrix whose i^{th} row $(i=1,2,\ldots,n)$ is composed of n terms of an $(AP)_{i-1}$ with constant of progression a_i. Then $|A|$ must be equal to $\prod_{i=1}^n a_i$.

Bicknell and Hoggatt refer to this as Eves' Theorem after a letter they received from Howard Eves; however, very similar results may be found much earlier in Muir and Metzler (see [10], pp. 47-48 and Ch. XX). The first example of such a matrix given in both of these sources is the familiar rectangular form of Pascal's triangle,

$$T=\begin{pmatrix}1&1&1&1&1&1&\cdots\\1&2&3&4&5&6&\cdots\\1&3&6&10&15&21&\cdots\\1&4&10&20&35&56&\cdots\\1&5&15&35&70&126&\cdots\\1&6&21&56&126&252&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix},\tag{1}$$

whose i^{th} row $(i=1,2,\ldots)$ is an $(AP)_{i-1}$ with constant 1. According to the theorem, then, the determinant of any $n\times n$ submatrix of T with one side on the left column of ones (or, by symmetry, its top row along the top row of ones) must equal $\prod_{i=1}^n a_i=\prod_{i=1}^n 1=1$.

An alternate approach involves the observation that

$$T=\begin{pmatrix}1&0&0&0&0&0&\cdots\\1&1&0&0&0&0&\cdots\\1&2&1&0&0&0&\cdots\\1&3&3&1&0&0&\cdots\\1&4&6&4&1&0&\cdots\\1&5&10&10&5&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&1&1&1&1&1&\cdots\\0&1&2&3&4&5&\cdots\\0&0&1&3&6&10&\cdots\\0&0&0&1&4&10&\cdots\\0&0&0&0&1&5&\cdots\\0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}; \tag{2}$$

that is, Pascal's triangle in rectangular form is equal to the matrix product of its lower triangular form with its upper triangular form.

From this decomposition, it is easy to see why the upper left corner determinants discussed above must equal one. Furthermore, it begs the question: can other arithmetic matrices be decomposed in a similar way?

The answer is yes. In fact, any matrix A whose rows are arithmetic progressions satisfying the criteria of Eves' theorem may be decomposed similarly. We state this formally as

Theorem 2 (Pascal Decomposition Theorem): Let any $n\times n$ matrix whose i^{th} row is an $(AP)_{i-1}$ for $i=1,2,\ldots,n$ be known as an arithmetic matrix. Then A is an arithmetic matrix if and only if it may be rewritten as the product of an $n\times n$ lower triangular seed matrix S with nonzero diagonal elements and the upper triangular matrix form of Pascal's triangle.

Proof: We will first present a constructive proof that such a matrix decomposes and then deal with the reverse case. Let

$$A=\begin{pmatrix}\overline{A}_1\\\overline{A}_2\\\vdots\\\overline{A}_n\end{pmatrix}, \tag{3}$$

where $\overline{A}_i$ is the i^{th} row of A, that is, $\overline{A}_i=(a_{i1},a_{i2},\ldots,a_{in})$, and let $\overline{A}_i$ be an $(AP)_{i-1}$ as defined above. We write out the difference table of this i^{th} row as in Sloane and Plouffe (see [12], p. 13), labeling the leading diagonal $\{b_{i1},b_{i2},\ldots\}$:

$$\begin{array}{lcccccccc}\overline{A}_i & b_{i1}=a_{i1} & & a_{i2} & & a_{i3} & a_{i4} & a_{i5} & a_{i6} \;\cdots\\ \Delta\overline{A}_i & & b_{i2} & & \cdot & \cdot & \cdot & \cdot & \cdot\\ \Delta^2\overline{A}_i & & & b_{i3} & & \cdot & \cdot & \cdot & \cdot\\ \vdots & & & & \ddots & \cdot & \cdot & \cdot & \\ \Delta^{i-1}\overline{A}_i & & & & b_{ii} & \cdot & \cdot & & \\ \Delta^{i}\overline{A}_i & & & & & 0 & \cdot & & \end{array}, \tag{4}$$

where $\Delta^k\overline{A}_i$ denotes the k^{th} row of $\overline{A}_i$'s differences; that is, the j^{th} element of $\Delta\overline{A}_i$ is $\Delta a_{ij}=a_{i(j+1)}-a_{ij}$ and, in general, the j^{th} element of $\Delta^k\overline{A}_i$ is $\Delta^k a_{ij}=\Delta^{k-1}a_{i(j+1)}-\Delta^{k-1}a_{ij}$.

Now, since $\overline{A}_i$ is an $(AP)_{i-1}$, its $(i-1)^{\text{th}}$ row of differences must be equal to the nonzero constant of the progression. In particular, b_{ii} must equal the constant of the progression. Also, any elements below row i (on the leading diagonal, all b_{ij}, $j>i$) must equal zero. From [12], we have the following relationships between the top row of our difference table and its leading diagonal:

$$a_{ik}=\sum_{j=1}^{k}\binom{k-1}{j-1}b_{ij} \quad\text{and}\quad b_{ik}=\sum_{j=1}^{k}(-1)^{k-j}\binom{k-1}{j-1}a_{ij}. \tag{5}$$

Substituting for the a_{ik} that make up our matrix A, we have

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}=\begin{pmatrix} \sum_{j=1}^{1}\binom{0}{j-1}b_{1j} & \cdots & \sum_{j=1}^{n}\binom{n-1}{j-1}b_{1j} \\ \sum_{j=1}^{1}\binom{0}{j-1}b_{2j} & \cdots & \sum_{j=1}^{n}\binom{n-1}{j-1}b_{2j} \\ \vdots & & \vdots \\ \sum_{j=1}^{1}\binom{0}{j-1}b_{nj} & \cdots & \sum_{j=1}^{n}\binom{n-1}{j-1}b_{nj} \end{pmatrix}. \tag{6}$$

Now, since some of the b_{ij} were shown to be zero above, we can reindex the sums and see that

$$A=\begin{pmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 & 1 & \cdots & \binom{n-1}{0} \\ 0 & 1 & 2 & \cdots & \binom{n-1}{1} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-2} \\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-1} \end{pmatrix}. \tag{7}$$

Therefore, A can be written as the product of a lower triangular matrix S and the $n\times n$ upper triangular Pascal matrix. Moreover, it is easy to see that $b_{ii}\neq 0$ for $(i=1,2,\ldots,n)$ by the definition of the $(AP)_i$.

As for the reverse case, we notice that so long as the diagonal elements of S are nonzero, the process outlined above can be run backwards. Hence, any matrix that is the product of a lower triangular seed matrix S with nonzero diagonal elements and the upper triangular matrix form of the Pascal triangle must be an (AP) matrix, and our theorem is proved. What's more, we now know the exact structure of the seed matrix S, and can calculate it from our original matrix A. We call this process the *Pascal decomposition* of A.

Corollary: $|A|=\prod_{i=1}^{n}b_{ii}$.

As an example, we can apply our theorem to the numbers $M_{k,r}$ examined by Wong and Maddocks in [13]. These numbers, with properties somewhat similar to binomial coefficients, satisfy the recurrence relation

$$M_{k+1,r+1}=M_{k+1,r}+M_{k,r+1}+M_{k,r} \tag{8}$$

with initial conditions $M_{0,0}=M_{1,0}=M_{0,1}=1$. If we write these numbers out in a matrix where k is the row number and r indicates the column, we have the following arithmetic matrix:

$$M=\begin{pmatrix}1&1&1&1&1&1&\cdots\\1&3&5&7&9&11&\cdots\\1&5&13&25&41&61&\cdots\\1&7&25&63&129&231&\cdots\\1&9&41&129&321&681&\cdots\\1&11&61&231&681&1683&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}. \tag{9}$$

To decompose M, we multiply it by the inverse of the upper triangular Pascal matrix. Equivalently, we could write out the difference tables for each row of M, but the inversion method is more succinct:

$$\begin{pmatrix}1&1&1&1&1&1&\cdots\\1&3&5&7&9&11&\cdots\\1&5&13&25&41&61&\cdots\\1&7&25&63&129&231&\cdots\\1&9&41&129&321&681&\cdots\\1&11&61&231&681&1683&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&1&1&1&1&1&\cdots\\0&1&2&3&4&5&\cdots\\0&0&1&3&6&10&\cdots\\0&0&0&1&4&10&\cdots\\0&0&0&0&1&5&\cdots\\0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}^{-1}$$
$$=\begin{pmatrix}1&0&0&0&0&0&\cdots\\1&2&0&0&0&0&\cdots\\1&4&4&0&0&0&\cdots\\1&6&12&8&0&0&\cdots\\1&8&24&32&16&0&\cdots\\1&10&40&80&80&32&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}, \tag{10}$$

which may be rewritten as

$$M=\begin{pmatrix}1&0&0&0&0&0&\cdots\\1&2&0&0&0&0&\cdots\\1&4&4&0&0&0&\cdots\\1&6&12&8&0&0&\cdots\\1&8&24&32&16&0&\cdots\\1&10&40&80&80&32&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&1&1&1&1&1&\cdots\\0&1&2&3&4&5&\cdots\\0&0&1&3&6&10&\cdots\\0&0&0&1&4&10&\cdots\\0&0&0&0&1&5&\cdots\\0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}. \tag{11}$$

From equation (11), it is easy to see that $|M|_{n\times n}=2^{n(n-1)/2}$, as predicted by Eves' theorem [just note that each row i ($i=1,\dots,n$) has constant 2^{i-1}].

Interestingly, symmetric matrices such as M are subject to further decomposition using the lower triangular matrix form of Pascal's triangle; note that

$$M=\begin{pmatrix}1&0&0&0&0&0&\cdots\\1&1&0&0&0&0&\cdots\\1&2&1&0&0&0&\cdots\\1&3&3&1&0&0&\cdots\\1&4&6&4&1&0&\cdots\\1&5&10&10&5&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0&0&0&\cdots\\0&2&0&0&0&0&\cdots\\0&0&4&0&0&0&\cdots\\0&0&0&8&0&0&\cdots\\0&0&0&0&16&0&\cdots\\0&0&0&0&0&32&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\cdot\begin{pmatrix}1&1&1&1&1&1&\cdots\\0&1&2&3&4&5&\cdots\\0&0&1&3&6&10&\cdots\\0&0&0&1&4&10&\cdots\\0&0&0&0&1&5&\cdots\\0&0&0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}. \tag{12}$$

This result is valid for symmetric arithmetic matrices in general, so we present another corollary.

Corollary: Let A be any symmetric matrix that also satisfies the conditions of Theorem 2. Then $A = P^T \cdot D \cdot P$, where P is the upper triangular Pascal triangle matrix and D is $diag(c_1, c_2, \ldots, c_n)$ with c_i being the constant of the progression for the i^{th} row of A $(i = 1, 2, \ldots, n)$.

Proof: By Theorem 2, $A = S_1 \cdot P$ and $A^T = P^T \cdot S_2$, where S_1 and S_2 are lower triangular and upper triangular matrices, respectively. Since A is symmetric and P is invertible, we can write $A = P^T \cdot L_1 \cdot P$ and $A^T = P^T \cdot L_2 \cdot P$, where L_1 is lower triangular and L_2 is upper triangular. Since $A = A^T$ by symmetry, $P^T \cdot L_1 \cdot P = P^T \cdot L_2 \cdot P$. Thus, $L_1 = L_2$, and since L_1 is lower triangular and L_2 is upper triangular, they must be a diagonal matrix, denoted by $diag(l_1, l_2, \ldots, l_n)$. Now, the diagonal elements of P and P^T are all one, and by the first corollary to Theorem 2 the determinant of the principal $(k \times k)$ submatrix of A is equal to the product of the progression constants of its rows, $c_1 c_2 \ldots c_k$. This means that $c_1 c_2 \ldots c_k = l_1 l_2 \ldots l_k$ for $k = 1, 2, \ldots, n$. Therefore, by induction on k, the diagonal elements of $D = L_1 = L_2$ must equal the progression constants for A's rows.

3. CONVOLUTION MATRICES FOR SEQUENCES WITH FIRST TERM ONE

The convolution matrices Bicknell and Hoggatt studied next provide further interesting examples of the decomposition technique, and they also lead to an interesting generalization. The *convolution* of two sequences $\{a_n\}$ and $\{b_n\}$ $(n = 0, 1, \ldots)$ is defined to be the sequence $\{c_n\}$ such that $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. The *convolution matrix* of a sequence is the matrix whose i^{th} column is the $(i-1)^{\text{th}}$ convolution of the sequence with itself $(i = 1, 2, \ldots)$. The rectangular form of Pascal's triangle, for instance, is the convolution matrix for the sequence $\{1, 1, 1, \ldots\}$. Bicknell and Hoggatt did a detailed analysis of the convolutions of the Catalan numbers

$$\{C_n\} = \left\{\frac{1}{n+1}\binom{2n}{n}\right\} = \{1, 1, 2, 5, 14, \ldots\}$$

over the course of several papers; in [2] and [3], they present the following convolution matrix for this sequence:

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 2 & 5 & 9 & 14 & 20 & 27 & \cdots \\ 5 & 14 & 28 & 48 & 75 & 110 & \cdots \\ 14 & 42 & 90 & 165 & 275 & 429 & \cdots \\ 42 & 132 & 297 & 572 & 1001 & 1638 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{13}$$

Bicknell and Hoggatt showed in [3] that any convolution matrix for a sequence whose first term is one must be an arithmetic progression matrix with row constants all equal to one and, hence, must—by Eves' theorem—have determinant one. Nevertheless, examining the Pascal decompositions for these matrices is worthwhile since it reveals a detailed underlying structure not otherwise apparent.

Looking at the Pascal decomposition of C, we note that the seed matrix S seems to have a close relationship to the even columns of C:

$$C=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 5 & 9 & 5 & 1 & 0 & 0 & \cdots \\ 14 & 28 & 20 & 7 & 1 & 0 & \cdots \\ 42 & 90 & 75 & 35 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & 0 & 1 & 3 & 6 & 10 & \cdots \\ 0 & 0 & 0 & 1 & 4 & 10 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{14}$$

What can account for this? To tease out the answer, we first examine convolution matrices in general. First, we note that any $n\times n$ convolution matrix V of a sequence $\{v_n\}$ may be written in the form

$$V=(\overline{V},\ A\cdot\overline{V},\ A^2\cdot\overline{V},\ldots,A^{n-1}\cdot\overline{V}), \tag{15}$$

where $\overline{V}$ is the first n terms of $\{v_n\}$ and

$$A=\begin{pmatrix} v_0 & 0 & 0 & \cdots & 0 \\ v_1 & v_0 & 0 & \cdots & 0 \\ v_2 & v_1 & v_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-1} & v_{n-2} & v_{n-3} & \cdots & v_0 \end{pmatrix}. \tag{16}$$

If we set $v_0=1$, then from [3] we know that V is a matrix satisfying Theorem 2 and must, therefore, have a Pascal decomposition, i.e., $V=S\cdot P$, where S is a lower triangular seed and P represents the upper triangular Pascal matrix. We can solve this for $S=V\cdot P^{-1}$; substituting for V gives

$$S=(\overline{V},\ A\cdot\overline{V},\ A^2\cdot\overline{V},\ldots,A^{n-1}\cdot\overline{V})\cdot P^{-1}. \tag{17}$$

Since the inverse of P is clearly

$$P^{-1}=\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & \cdots \\ 0 & 1 & -2 & 3 & -4 & 5 & \cdots \\ 0 & 0 & 1 & -3 & 6 & -10 & \cdots \\ 0 & 0 & 0 & 1 & -4 & 10 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -5 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{18}$$

we can rewrite S:

$$S=(\overline{V},\ (A-I)\cdot\overline{V},\ (A-I)^2\cdot\overline{V},\ldots,(A-I)^{n-1}\cdot\overline{V}), \tag{19}$$

where I is the identity matrix.

Thus, each column of S is a successive convolution of $\{v_0,v_1,\ldots,v_n\}$ with $\{0,v_1,v_2,\ldots,v_n\}$; i.e., if

$$B=(A-I)=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ v_1 & 0 & 0 & \cdots & 0 \\ v_2 & v_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ v_{n-1} & v_{n-2} & \cdots & v_1 & 0 \end{pmatrix}. \tag{20}$$

then

$$S=(\overline{V},\ B\cdot\overline{V},\ B^2\cdot\overline{V},\ldots,B^{n-1}\cdot\overline{V}). \tag{21}$$

We summarize this discussion in the following theorem.

Theorem 3 (Weak Convolution Decomposition Theorem): If V is a convolution matrix of a sequence $\{v_n\}$ with first term one, then $V = S \cdot P$ for some lower triangular matrix S and the upper triangular Pascal triangle matrix P. Moreover, successive columns of S are successive convolutions of the sequence $\{v_n\}$ with the sequence $\{0, v_1, v_2, v_3, \ldots\}$.

Returning to our Catalan convolution matrix example, we can reexamine its seed matrix in light of this theorem. As predicted, each column is a convolution of the sequences $\{1, 1, 2, 5, \ldots\}$ and $\{0, 1, 2, 5, \ldots\}$. Besides this, we can make our earlier conjecture about the relationship between the columns of the Catalan seed matrix, denoted by S_C, and the even columns of C explicit: the i^{th} column of S_C is equal to the $(2i)^{\text{th}}$ column of C shifted down i places ($i = 0, 1, \ldots$).

Symbolically, we let $C = (\overline{C},\ A \cdot \overline{C},\ A^2 \cdot \overline{C}, \ldots, A^{n-1} \cdot \overline{C})$, where $\overline{C}$ is the column vector filled with the first n Catalan numbers and

$$A = \begin{pmatrix} C_0 & 0 & 0 & \cdots & 0 \\ C_1 & C_0 & 0 & \cdots & 0 \\ C_2 & C_1 & C_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & C_{n-3} & \cdots & C_0 \end{pmatrix}. \tag{22}$$

Also, let $S_C = (\overline{C},\ B \cdot \overline{C},\ B^2 \cdot \overline{C}, \ldots, B^{n-1} \cdot \overline{C})$, where

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ C_1 & 0 & 0 & \cdots & 0 \\ C_2 & C_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & \cdots & C_1 & 0 \end{pmatrix}. \tag{23}$$

Then what we are trying to show is that $B^k \cdot \overline{C}$ is equal to $A^{2k} \cdot \overline{C}$ shifted down k spots.

We first note that the Catalan numbers have the well-known recursive relation $\sum_{j=0}^{i} C_{i-j} C_j = C_{i+1}$ for $i = 0, 1, \ldots$. (See [8].)

By this relation, we have

$$A^2 = \begin{pmatrix} C_1 & 0 & 0 & \cdots & 0 \\ C_2 & C_1 & 0 & \cdots & 0 \\ C_3 & C_2 & C_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n-1} & C_{n-2} & \cdots & C_1 \end{pmatrix} \tag{24}$$

and

$$I_S \cdot A^2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ C_1 & 0 & 0 & \cdots & 0 \\ C_2 & C_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & \cdots & C_1 & 0 \end{pmatrix} = B, \tag{25}$$

where

$$I_S = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{26}$$

Since $I_S \cdot A^2 = B$, we can show by mathematical induction that

$$\begin{aligned} S_C &= (\overline{C},\ B \cdot \overline{C},\ B^2 \cdot \overline{C}, \ldots, B^{n-1} \cdot \overline{C}) \\ &= (\overline{C}, (I_S \cdot A^2) \cdot \overline{C}, (I_S^2 \cdot A^4) \cdot \overline{C}, \ldots, (I_S^{n-1} \cdot A^{2(n-1)})\overline{C}), \end{aligned} \tag{27}$$

thereby showing the desired relationship between the columns of S_C and the even columns of C.

4. CONVOLUTION MATRICES OF SEQUENCES WITH FIRST TERM OTHER THAN ONE

We now have a very detailed understanding of the structure of any convolution matrix of a sequence whose first term is one. What happens, though, if the sequence's first term does not equal one? A good example is the following convolution matrix of the Lucas numbers {2, 1, 3, 4, 7, ...} (we use the standard definition and notation, but begin with $L_0 = 2$ instead of $L_1 = 1$):

$$L = \begin{pmatrix} 2 & 4 & 8 & 16 & 32 & 64 & \cdots \\ 1 & 4 & 12 & 32 & 80 & 192 & \cdots \\ 3 & 13 & 42 & 120 & 320 & 816 & \cdots \\ 4 & 22 & 85 & 280 & 840 & 2368 & \cdots \\ 7 & 45 & 195 & 705 & 2290 & 6924 & \cdots \\ 11 & 82 & 399 & 1588 & 5601 & 10204 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{28}$$

Eves' theorem has nothing to say in this case since the rows are no longer arithmetic progressions. However, if we multiply it twice by the inverse of the upper triangular Pascal triangle matrix, which we will again denote P, we obtain a seed matrix very like the ones encountered in our earlier work:

$$L \cdot (P^{-1})^2 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 7 & 2 & 0 & 0 & 0 & \cdots \\ 4 & 14 & 13 & 2 & 0 & 0 & \cdots \\ 7 & 31 & 43 & 19 & 2 & 0 & \cdots \\ 11 & 60 & 115 & 90 & 25 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{29}$$

In particular, each column of this matrix is equal to the convolution of the sequences {2, 1, 3, 4, ...} and {0, 1, 3, 4, ...}.

Note that this sequence had first term two, and that we multiplied the matrix by P^{-1} twice. This was by no means coincidental; in fact, we may state this correlation as part of a general theorem.

Theorem 4 (Strong Convolution Decomposition Theorem): Let $\{v_n\}$ be a sequence whose first term is a positive integer v_0, and let V be the convolution matrix of that sequence. Then $V = S \cdot P^{v_0}$ for some lower triangular matrix S and the upper triangular Pascal triangle matrix P. Moreover, successive columns of S are successive convolutions of the sequence $\{v_n\}$ with the sequence $\{0, v_1, v_2, v_3, \ldots\}$.

Proof: The proof is constructive. We first note that if V is any convolution matrix of a sequence $\{v_n\}$ with first term v_0, then $V = (\overline{V},\ A \cdot \overline{V},\ A^2 \cdot \overline{V}, \ldots, A^{n-1} \cdot \overline{V})$, where $\overline{V}$ is the column vector whose i^{th} element is v_i and

$$A=\begin{pmatrix} v_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ v_1 & v_0 & 0 & 0 & 0 & 0 & \cdots \\ v_2 & v_1 & v_0 & 0 & 0 & 0 & \cdots \\ v_3 & v_2 & v_1 & v_0 & 0 & 0 & \cdots \\ v_4 & v_3 & v_2 & v_1 & v_0 & 0 & \cdots \\ v_5 & v_4 & v_3 & v_2 & v_1 & v_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{30}$$

If we multiply the convolution matrix v_0 times by the inverse of the upper triangular Pascal triangle matrix P, we have

$$\begin{aligned} S &= V\cdot(P^{-1})^{v_0} \\ &= (\overline{V},\ A\cdot\overline{V},\ A^2\cdot\overline{V},\ldots,\ A^{n-1}\cdot\overline{V})\cdot(P^{-1})^{v_0} \\ &= (\overline{V},\ (A-I)\cdot\overline{V},\ (A-I)^2\cdot\overline{V},\ldots,(A-I)^{n-1}\cdot\overline{V})\cdot(P^{-1})^{v_0-1} \\ &= (\overline{V},\ (A-2I)\cdot\overline{V},\ (A-2I)^2\cdot\overline{V},\ldots,(A-2I)^{n-1}\cdot\overline{V})\cdot(P^{-1})^{v_0-2} \\ &\ \ \vdots \\ &= (\overline{V},\ (A-v_0I)\cdot\overline{V},\ (A-v_0I)^2\cdot\overline{V},\ldots,(A-v_0I)^{n-1}\cdot\overline{V}). \end{aligned} \tag{31}$$

Let a new matrix $B=A-v_0I$, i.e.,

$$B=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ v_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ v_2 & v_1 & 0 & 0 & 0 & 0 & \cdots \\ v_3 & v_2 & v_1 & 0 & 0 & 0 & \cdots \\ v_4 & v_3 & v_2 & v_1 & 0 & 0 & \cdots \\ v_5 & v_4 & v_3 & v_2 & v_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{32}$$

Then it is clear that S is a convolution matrix since $S=(\overline{V},\ B\cdot\overline{V},\ B^2\cdot\overline{V},\ldots,\ B^{n-1}\cdot\overline{V})$. More specifically, successive columns of S are successive convolutions of the sequence $\{v_0, v_1, v_2, \ldots\}$ with $\{0, v_1, v_2, \ldots\}$, as was to be shown.

Corollary: For any convolution matrix V satisfying the conditions of Theorem 4, $|V| = v_0^n v_1^{n(n-1)/2}$.

Proof: By Theorem 4, $V=S\cdot P^{v_0}$. Now, $|P^{v_0}|=1^{v_0}=1$, so $|V|=|S|$. Since S is lower triangular with diagonal elements $v_0, v_0v_1^1, v_0v_1^2, \ldots, v_0v_1^{n-1}$, $|S|=v_0^n v_1^{1+2+\cdots+(n-1)}$. Hence, $|V|=|S|=v_0^n v_1^{n(n-1)/2}$.

Remark: The determinant of any convolution matrix is wholly determined by the first two elements of the sequence.

5. CONCLUSION AND FUTURE GOALS

Pascal decompositions allow easy calculation of determinants for arbitrary sized matrices, for once the sequence on the diagonal of the seed matrix is understood, it is a simple matter to calculate its product. What's more, this technique provides a visual tool to examine the structure of several flavors of matrices, such as the arithmetic and convolution matrices discussed above.

In a future paper, we hope to further generalize this technique and add to this list the recursion relation matrices studied by Ollerton and Shannon [11].

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REFERENCES

1. M. Bicknell. "A Primer for the Fibonacci Numbers: Part XIII. The Fibonacci Convolution Triangle, Pascal's Triangle, and Some Interesting Determinants." *The Fibonacci Quarterly* **11.5** (1973):511-16.
2. M. Bicknell & V. E. Hoggatt, Jr. "Catalan and Related Sequences Arising from Inverses of Pascal's Triangle Matrices." *The Fibonacci Quarterly* **14.5** (1976):395-405.
3. M. Bicknell & V. E. Hoggatt, Jr. "Pascal, Catalan, and General Sequence Convolution Arrays in a Matrix." *The Fibonacci Quarterly* **14.2** (1976):135-43.
4. M. Bicknell & V. E. Hoggatt, Jr. "Sequences of matrix Inverses from Pascal, Catalan, and Related Convolution Arrays." *The Fibonacci Quarterly* **14.3** (1976):224-32.
5. M. Bicknell & V. E. Hoggatt, Jr. "Special Determinants Found within Generalized Pascal Triangles." *The Fibonacci Quarterly* **11.5** (1973):469-79.
6. M. Bicknell & V. E. Hoggatt, Jr. "Unit Determinants in Generalized Pascal Triangles." *The Fibonacci Quarterly* **11.2** (1973):131-44.
7. B. A. Bondarenko. *Generalized Pascal Triangles and Pyramids*. Santa Clara, CA: The Fibonacci Association, 1993.
8. D. Callan. "A Combinatorial Interpretation of a Catalan Numbers Identity." *Math. Mag.* **72.4** (1999):295-98.
9. V. E. Hoggatt, Jr., & G. E. Bergum. "Generalized Convolution Arrays." *The Fibonacci Quarterly* **13.3** (1975):193-98.
10. T. Muir & W. H. Metzler. *A Treatise on the Theory of Determinants*. New York: Longmans, Green and Co., 1933.
11. R. L. Ollerton & A. G. Shannon. "Some Properties of Generalized Pascal Squares and Triangles." *The Fibonacci Quarterly* **36.2** (1998):98-109.
12. N. J. A. Sloane & S. Plouffe. *The Encyclopedia of Integer Sequences*. San Diego, CA: Academic Press, 1995.
13. C. K. Wong & T. W. Maddocks. "A Generalized Pascal's Triangle." *The Fibonacci Quarterly* **13.2** (1975):134-36.

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