ON THE SUMMATION OF GENERALIZED ARITHMETIC-GEOMETRIC TRIGONOMETRIC SERIES

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1. INTRODUCTION

For any real or complex number β, we denote

$$(t\,|\,\beta)_p = \prod_{j=0}^{p-1}(t-j\beta),$$

where p is a positive integer, with $(t\,|\,\beta)_0 = 1$, and call it the generalized falling factorial with increment β. In particular, we write $(t\,|\,1)_p = (t)_p$ and $(t\,|\,0)_p = t^p$. It is known that the Dickson polynomial in t of degree p with real parameter α is defined as

$$D_p(t,\alpha) = \sum_{i=0}^{[p/2]} \frac{p}{p-i}\binom{p-i}{i}(-\alpha)^i t^{p-2i} \tag{1.1}$$

with $D_0(t,\alpha) = 2$ (cf. [4]). Evidently $D_p(t,0) = t^p$.

In this paper, we find closed summation formulas for the series

$$S_1^{(1)}(n) = \sum_{k=0}^{n}(k+\lambda\,|\,\beta)_p r^k \cos k\theta, \quad S_1^{(1)}(\infty) = \sum_{k=0}^{\infty}(k+\lambda\,|\,\beta)_p r^k \cos k\theta, \tag{1.2}$$

$$S_1^{(2)}(n) = \sum_{k=0}^{n}(k+\lambda\,|\,\beta)_p r^k \sin k\theta, \quad S_1^{(2)}(\infty) = \sum_{k=0}^{\infty}(k+\lambda\,|\,\beta)_p r^k \sin k\theta, \tag{1.3}$$

$$S_2^{(1)}(n) = \sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha) r^k \cos k\theta, \quad S_2^{(1)}(\infty) = \sum_{k=\alpha}^{\infty} D_p(k,\alpha) r^k \cos k\theta, \tag{1.4}$$

$$S_2^{(2)}(n) = \sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha) r^k \sin k\theta, \quad S_2^{(2)}(\infty) = \sum_{k=\alpha}^{\infty} D_p(k,\alpha) r^k \sin k\theta, \tag{1.5}$$

where α is any given integer, λ and β are real numbers, and $|r| < 1$ for $S_i^{(j)}(\infty)$, $i, j = 1, 2$.

In [2], L. C. Hsu and P. J. S. Shiue have obtained closed summation formulas for the series

$$S_1(n) = \sum_{k=0}^{n}(k+\lambda\,|\,\beta)_p x^k, \quad S_1(\infty) = \sum_{k=0}^{\infty}(k+\lambda\,|\,\beta)_p x^k, \tag{1.6}$$

$$S_2(n) = \sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha) x^k, \quad S_2(\infty) = \sum_{k=\alpha}^{\infty} D_p(k,\alpha) x^k, \tag{1.7}$$

where α is any given integer, λ and β are real or complex numbers, and $|x| < 1$ for both $S_1(\infty)$ and $S_2(\infty)$. The results of this paper are based on the conclusions above.

2. MAIN RESULTS

We first define the rank as follows.

Definition 2.1: The number of the summation symbols $\sum$ appearing in the right-hand side of a closed summation formula is called the *rank* of this summation formula.

Recall Howard's degenerate weighted Stirling numbers $S(p, j, \lambda|\beta)$ $(0 \le j \le p)$ can be defined by the basis transformation relation

$$(t+\lambda|\beta)_p = \sum_{j=0}^{p} j!S(p, j, \lambda|\beta)\binom{t}{j}. \tag{2.1}$$

Indeed, by applying the forward difference operator Δ defined by $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^j = \Delta\Delta^{j-1}$ $(j \ge 2)$, and using the Newton interpolation formula to the LHS of (2.1), we see that the numbers $j!S(p, j, \lambda|\beta)$ in the RHS of (2.1) may be written as (cf. [2])

$$j!S(p, j, \lambda|\beta) = \Delta^j(t+\lambda|\beta)_{p|t=0} = \sum_{m=0}^{j}(-1)^{j-m}\binom{j}{m}(m+\lambda|\beta)_p. \tag{2.2}$$

Equation (2.2) shows the rank of $S(p, j, \lambda|\beta)$ is 1.

On the other hand, a kind of generalized Stirling numbers, called Dickson Stirling numbers, can be introduced by the relations (cf. [1], [2])

$$D_p(t, \alpha) = \sum_{j=0}^{p} S(p, j, \alpha)(t-\alpha)_j \quad (p = 1, 2, \ldots). \tag{2.3}$$

Of course, these relations may be rewritten as follows:

$$D_p(t+\alpha, \alpha) = \sum_{j=0}^{p} S(p, j, \alpha)(t)_j \quad (p = 1, 2, \ldots). \tag{2.4}$$

In fact, similar to the expression of $S(p, j, \lambda|\beta)$, the Dickson-Stirling numbers have the finite difference expression

$$S(p, j, \alpha) = \frac{1}{j!}\Delta^j D_p(t, \alpha)|_{t=\alpha}$$

and its rank is 1.

In the following, we first list the main results of L. C. Hsu and P. J. S. Shiue (cf. [2]) which are important to our conclusions. Denote

$$\phi(x, n, j) = \frac{1}{1-x}\left[\left(\frac{x}{1-x}\right)^j - x^{n+1}\sum_{r=0}^{j}\binom{n+1}{j-r}\left(\frac{x}{1-x}\right)^r\right]. \tag{2.5}$$

Lemma 2.1: For $x \ne 1$, we have the summation formula

$$\sum_{k=0}^{n}(k+\lambda|\beta)_p x^k = \sum_{j=0}^{p} j!S(p, j, \lambda|\beta)\phi(x, n, j), \tag{2.6}$$

where $\phi(x, n, j)$ is given by (2.5).

Lemma 2.2: For $|x| < 1$, we have the summation formula

$$\sum_{k=0}^{\infty}(k+\lambda|\beta)_p x^k = \sum_{j=0}^{p}\frac{j!S(p,j,\lambda|\beta)x^j}{(1-x)^{j+1}}. \tag{2.7}$$

Lemma 2.3: For any given integer α, we have the summation formula

$$\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha)x^k = x^\alpha \sum_{j=0}^{p} j!S(p,j,\alpha)\phi(x,n,j), \tag{2.8}$$

where the Dickson-Stirling numbers are defined by (2.3) and $\phi(x,n,j)$ is given by (2.5).

Lemma 2.4: For $|x|<1$ and any given integer α, we have the formula

$$\sum_{k=\alpha}^{\infty} D_p(k,\alpha)x^k = x^\alpha \sum_{j=0}^{p}\frac{j!S(p,j,\alpha)x^j}{(1-x)^{j+1}}. \tag{2.9}$$

The proofs of Lemmas 2.1 through 2.4 can be seen in [2].

In these lemmas, the constants λ, β are real or complex numbers, α is an integer. From now on, unless specified, we assume λ, β are real parameters, α is an integer.

We first recall the famous Chebyshev polynomial $T_n(x)$ defined as follows:

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1,1].$$

It is known that $T_n(x)$ satisfies the recurrence relations $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ with $T_0(x)=1$, $T_1(x)=x$. For simplicity, denote

$$\begin{aligned} \cos t_1 &= \frac{\cos\theta - r}{\sqrt{1+r^2-2r\cos\theta}}, \\ \sin t_1 &= \frac{\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}, \\ T_l &= T_l\left(\frac{\cos\theta - r}{\sqrt{1+r^2-2r\cos\theta}}\right), \end{aligned} \tag{2.10}$$

where $T_l(x)$ is the Chebyshev polynomial of degree l.

Theorem 2.1: Assume $0 \le \theta \le 2\pi$, $r \ge 0$. If $r \ne 1$ or $\theta \ne 0, 2\pi$, we have the summation formulas

$$\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k \cos k\theta = \sum_{j=0}^{p} j!S(p,j,\lambda|\beta)\phi_1^{(1)}(r,\theta,n,j), \tag{2.11}$$

$$\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k \sin k\theta = \sum_{j=0}^{p} j!S(p,j,\lambda|\beta)\phi_1^{(2)}(r,\theta,n,j), \tag{2.12}$$

where

$$\phi_1^{(1)}(r,\theta,n,j) = r^j\left[\frac{(1-r\cos\theta)T_j}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} - \frac{r\sin^2\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_l T_1^{j-1-l}\right]$$

$$-r^{n+1}\sum_{m=0}^{j}\binom{n+1}{j-m}r^m\left\{\frac{T_m[\cos(n+1)\theta - r\cos n\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+2}{2}}} - \frac{\sin\theta\sin(n+1)\theta - r\sin\theta\sin n\theta}{(1+r^2-2r\cos\theta)^{\frac{m+3}{2}}}\sum_{l=0}^{m-1}T_l T_1^{m-1-l}\right\},$$

$$\phi_1^{(2)}(r,\theta,n,j)=r^j\left[\frac{r\sin\theta T_j}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}}+\frac{\sin\theta(1-r\cos\theta)}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right]$$

$$-r^{n+1}\sum_{m=0}^{j}\binom{n+1}{j-m}r^m\left\{\frac{T_m[\sin(n+1)\theta-r\sin n\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+2}{2}}}+\frac{\sin\theta\cos(n+1)\theta-r\sin\theta\cos n\theta}{(1+r^2-2r\cos\theta)^{\frac{m+3}{2}}}\sum_{l=0}^{m-1}T_lT_1^{m-1-l}\right\}.$$

Equations (2.11) and (2.12) imply that the summation formulas of $\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k\cos k\theta$ and $\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k\sin k\theta$ have the same rank 5.

Proof: In (2.6), set $x=re^{1\theta}$, then

$$\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k\cos k\theta=\sum_{j=0}^{p}j!S(p,j,\lambda|\beta)\operatorname{Re}\phi(re^{i\theta},n,j),\tag{2.13}$$

$$\sum_{k=0}^{n}(k+\lambda|\beta)_p r^k\sin k\theta=\sum_{j=0}^{p}j!S(p,j,\lambda|\beta)\operatorname{Im}\phi(re^{i\theta},n,j).\tag{2.14}$$

We first obtain

$$\begin{aligned}\frac{x}{1-x}&=\frac{re^{i\theta}}{1-re^{i\theta}}=\frac{r(\cos\theta+i\sin\theta)}{1-r\cos\theta-ir\sin\theta}=\frac{r(\cos\theta-r+i\sin\theta)}{1+r^2-2r\cos\theta}\\&=\frac{r\sqrt{1+r^2-2r\cos\theta}}{1+r^2-2r\cos\theta}\left(\frac{\cos\theta-r}{\sqrt{1+r^2-2r\cos\theta}}+\frac{i\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}\right)\\&=\frac{r}{\sqrt{1+r^2-2r\cos\theta}}(\cos t_1+i\sin t_1)=\frac{re^{it}1}{\sqrt{1+r^2-2r\cos\theta}},\end{aligned}\tag{2.15}$$

where $\cos t_1$ and $\sin t_1$ are defined in (2.10), and

$$\begin{aligned}\frac{1}{1-x}&=\frac{1}{1-re^{i\theta}}=\frac{1}{\sqrt{1+r^2-2r\cos\theta}}\left(\frac{1-r\cos\theta}{\sqrt{1+r^2-2r\cos\theta}}+i\frac{r\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}\right)\\&=\frac{e^{it}2}{\sqrt{1+r^2-2r\cos\theta}},\end{aligned}\tag{2.16}$$

where

$$\cos t_2=\frac{1-r\cos\theta}{\sqrt{1+r^2-2r\cos\theta}},\quad \sin t_2=\frac{r\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}.\tag{2.17}$$

Therefore,

$$\begin{aligned}&\phi(x,n,j)=\phi(re^{i\theta},n,j)\\&=\frac{e^{it}2}{\sqrt{1+r^2-2r\cos\theta}}\left[\frac{r^je^{jt}1^i}{(1+r^2-2r\cos\theta)^{\frac{j}{2}}}-r^{n+1}e^{i(n+1)\theta}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^me^{mt}1^i}{(1+r^2-2r\cos\theta)^{\frac{m}{2}}}\right]\\&=\frac{r^je^{(jt_1+t_2)i}}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}}-r^{n+1}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^me^{[mt_1+t_2+(n+1)\theta]i}}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}}.\end{aligned}\tag{2.18}$$

Equation (2.18) implies that

$$\operatorname{Re}\phi(re^{i\theta},n,j)=\frac{r^j\cos(jt_1+t_2)}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}}-r^{n+1}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m\cos[mt_1+t_2+(n+1)\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}},\tag{2.19}$$

$$\operatorname{Im}\phi(re^{i\theta},n,j)=\frac{r^j\sin(jt_1+t_2)}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}}-r^{n+1}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m\sin[mt_1+t_2+(n+1)\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}}.\tag{2.20}$$

By the definition of Chebyshev polynomial and (2.10), we know $T_l=\cos(lt_1)$. Set $I_j=\sin jt_1$, then

$$\begin{aligned}I_j &=\sin jt_1=\sin(j-1)t_1\cos t_1+\cos(j-1)t_1\sin t_1\\ &=I_{j-1}\cos t_1+T_{j-1}\sin t_1=(I_{j-2}\cos t_1+T_{j-2}\sin t_1)\cos t_1+T_{j-1}\sin t_1\\ &=(I_{j-3}\cos t_1+T_{j-3}\sin t_1)\cos^2 t_1+T_{j-2}\sin t_1\cos t_1+T_{j-1}\sin t_1\\ &=I_{j-3}\cos^3 t_1+\sin t_1(T_{j-3}\cos^2 t_1+T_{j-2}\cos t_1+T_{j-1})\\ &\dots\\ &=I_1\cos^{j-1}t_1+\sin t_1(T_1\cos^{j-2}t_1+T_2\cos^{j-3}t_1+\cdots+T_{j-2}\cos t_1+T_{j-1})\\ &=\sin t_1(\cos^{j-1}t_1+T_1\cos^{j-2}t_1+T_2\cos^{j-3}t_1+\cdots+T_{j-2}\cos t_1+T_{j-1})\\ &=\sin t_1\sum_{l=0}^{j-1}T_l t_1^{j-1-l}.\end{aligned}\tag{2.21}$$

From (2.21) and (2.17), it is easy to obtain that

$$\begin{aligned}\cos(jt_1+t_2)&=\cos jt_1\cos t_2-\sin jt_1\sin t_2=T_j\cos t_2-\sin t_1\sin t_2\sum_{l=0}^{j-1}T_l t_1^{j-1-l}\\ &=\frac{(1-r\cos\theta)T_j}{\sqrt{1+r^2-2r\cos\theta}}-\frac{r\sin^2\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}.\end{aligned}\tag{2.22}$$

Similarly,

$$\begin{aligned}\sin(jt_1+t_2)&=\sin jt_1\cos t_2+\cos jt_1\sin t_2\\ &=\frac{\sin\theta(1-r\cos\theta)}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}+\frac{rT_j\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}.\end{aligned}\tag{2.23}$$

Hence,

$$\begin{aligned}\cos(jt_1+t_2+\alpha\theta)&=\cos(jt_1+t_2)\cos\alpha\theta-\sin(jt_1+t_2)\sin\alpha\theta\\ &=\left[\frac{(1-r\cos\theta)T_j}{\sqrt{1+r^2-2r\cos\theta}}-\frac{r\sin^2\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right]\cos\alpha\theta\\ &\quad-\left[\frac{\sin\theta(1-r\cos\theta)}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}+\frac{rT_j\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}\right]\sin\alpha\theta\\ &=\frac{T_j(\cos\alpha\theta-r\cos\theta\cos\alpha\theta-r\sin\theta\sin\alpha\theta)}{\sqrt{1+r^2-2r\cos\theta}}\\ &\quad-\frac{r\sin^2\theta\cos\alpha\theta+\sin\theta\sin\alpha\theta-r\sin\theta\cos\theta\sin\alpha\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\\ &=\frac{T_j[\cos\alpha\theta-r\cos(1-\alpha)\theta]}{\sqrt{1+r^2-2r\cos\theta}}-\frac{\sin\theta\sin\alpha\theta+4\sin\theta\sin(1-\alpha)\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l},\end{aligned}\tag{2.24}$$

$$\begin{aligned}\sin(jt_1+t_2+\alpha\theta) &= \sin(jt_1+t_2)\cos\alpha\theta+\cos(jt_1+t_2)\sin\alpha\theta\\ &=\left[\frac{\sin\theta(1-r\cos\theta)}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}+\frac{rT_j\sin\theta}{\sqrt{1+r^2-2r\cos\theta}}\right]\cos\alpha\theta\\ &\quad+\left[\frac{(1-r\cos\theta)T_j}{\sqrt{1+r^2-2r\cos\theta}}-\frac{r\sin^2\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right]\sin\alpha\theta\\ &=\frac{T_j[\sin\alpha\theta+r\sin(1-\alpha)\theta]}{\sqrt{1+r^2-2r\cos\theta}}+\frac{\sin\theta\cos\alpha\theta-r\sin\theta\cos(1-\alpha)\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}.\end{aligned}\tag{2.25}$$

In (2.24) and (2.25), set $j=m$ and $\alpha=n+1$ to obtain

$$\begin{aligned}\cos[mt_1+t_2+(n+1)\theta]&=\frac{T_m[\cos(n+1)\theta-r\cos n\theta]}{\sqrt{1+r^2-2r\cos\theta}}\\ &\quad-\frac{\sin\theta\sin(n+1)\theta-r\sin\theta\sin n\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{m-1}T_lT_1^{m-1-l},\end{aligned}\tag{2.26}$$

$$\begin{aligned}\sin[mt_1+t_2+(n+1)\theta]&=\frac{T_m[\sin(n+1)\theta-r\sin n\theta]}{\sqrt{1+r^2-2r\cos\theta}}\\ &\quad+\frac{\sin\theta\cos(n+1)\theta-r\sin\theta\cos n\theta}{1+r^2-2r\cos\theta}\sum_{l=0}^{m-1}T_lT_1^{m-1-l}.\end{aligned}\tag{2.27}$$

From (2.13), (2.14), (2.19), (2.20), (2.22), (2.23), (2.26), and (2.27), we obtain (2.11) and (2.12) immediately.

In (2.11) and (2.12), set $n\to\infty$ to obtain the following conclusion.

Theorem 2.2: If $r<1$ and $0\le\theta\le 2\pi$, then

$$\sum_{k=0}^{\infty}(k+\lambda|\beta)_p r^k\cos k\theta=\sum_{j=0}^{p}j!S(p,j,\lambda|\beta)\psi_1^{(1)}(r,\theta,j)\tag{2.28}$$

and

$$\sum_{k=0}^{\infty}(k+\lambda|\beta)_p r^k\sin k\theta=\sum_{j=0}^{p}j!S(p,j,\lambda|\beta)\psi_1^{(2)}(r,\theta,j),\tag{2.29}$$

where

$$\psi_1^{(1)}(r,\theta,j)=r^j\left[\frac{T_j(1-r\cos\theta)}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}}-\frac{r\sin^2\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right]$$

and

$$\psi_1^{(2)}(r,\theta,j)=r^j\left[\frac{rT_j\sin\theta}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}}+\frac{\sin\theta(1-r\cos\theta)}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right].$$

The rank of (2.28) and (2.29) is 3.

Theorem 2.3: Assume $r\ge 0$, $0\le\theta\le 2\pi$. If $r\ne 1$ or $\theta\ne 0, 2\pi$ for any given integer α, we have the summation formulas:

$$\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha)r^k \cos k\theta = \sum_{j=0}^{p} j!S(p,j,\alpha)\phi_2^{(1)}(\alpha,r,\theta,n,j), \tag{2.30}$$

$$\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha)r^k \sin k\theta = \sum_{j=0}^{p} j!S(p,j,\alpha)\phi_2^{(2)}(\alpha,r,\theta,n,j), \tag{2.31}$$

where

$$\begin{aligned}\phi_2^{(1)}(\alpha,r,\theta,n,j) = r^{j+\alpha}&\left[T_j\frac{\cos\alpha\theta - r\cos(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} - \frac{\sin\theta\sin\alpha\theta + r\sin\theta\sin(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right]\\ &-r^{n+1+\alpha}\sum_{m=0}^{j}\binom{n+1}{j-m}r^m\left[T_m\frac{\cos(n+1+\alpha)\theta - r\cos(n+\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{m+2}{2}}}\right.\\ &\left.-\frac{\sin\theta\sin(n+1+\alpha)\theta - r\sin\theta\sin(n+\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{m+3}{2}}}\sum_{l=0}^{m-1}T_lT_1^{m-1-l}\right],\end{aligned}$$

$$\begin{aligned}\phi_2^{(2)}(\alpha,r,\theta,n,j) = r^{j+\alpha}&\left[T_j\frac{\sin\alpha\theta - r\sin(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} + \frac{\sin\theta\cos\alpha\theta - r\sin\theta\cos(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right]\\ &-r^{n+1+\alpha}\sum_{m=0}^{j}\binom{n+1}{j-m}r^m\left[T_m\frac{\sin(n+1+\alpha)\theta - r\sin(n+\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{m+2}{2}}}\right.\\ &\left.+\frac{\sin\theta\cos(n+1+\alpha)\theta - r\sin\theta\cos(n+\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{m+3}{2}}}\sum_{l=0}^{m-1}T_lT_1^{m-1-l}\right].\end{aligned}$$

Equations (2.30) and (2.31) imply that the summation formulas of $\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha)r^k\cos k\theta$ and $\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha)r^k\sin k\theta$ have the same rank 5.

Proof: In (2.8), set $x = re^{1\theta}$, then

$$\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha)r^k \cos k\theta = \sum_{j=0}^{p} j!S(p,j,\alpha)\,\mathrm{Re}[r^\alpha e^{i\alpha\theta}\phi(re^{i\theta},n,j)], \tag{2.32}$$

$$\sum_{k=\alpha}^{n+\alpha} D_p(k,\alpha)r^k \sin k\theta = \sum_{j=0}^{p} j!S(p,j,\alpha)\,\mathrm{Im}[r^\alpha e^{i\alpha\theta}\phi(re^{i\theta},n,j)]. \tag{2.33}$$

By (2.18), we have

$$\begin{aligned}x^\alpha\phi(x,n,j) &= r^\alpha e^{i\alpha\theta}\phi(re^{i\theta})\\ &= \frac{r^{j+\alpha}e^{(jt_1+t_2+\alpha\theta)i}}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}} - r^{n+1+\alpha}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m e^{[mt_1+t_2+(n+1+\alpha)\theta]i}}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}}.\end{aligned} \tag{2.34}$$

This implies

$$\begin{aligned}\mathrm{Re}[r^\alpha e^{i\alpha\theta}\phi(re^{i\theta},n,j)] &= \frac{r^{j+\alpha}\cos(jt_1+t_2+\alpha\theta)}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}}\\ &\quad - r^{n+1+\alpha}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m\cos[mt_1+t_2+(n+1+\alpha)\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}},\end{aligned} \tag{2.35}$$

$$\operatorname{Im}[r^{\alpha}e^{i\alpha\theta}\phi(re^{i\theta},n,j)] = \frac{r^{j+\alpha}\sin(jt_1+t_2+\alpha\theta)}{(1+r^2-2r\cos\theta)^{\frac{j+1}{2}}} - r^{n+1+\alpha}\sum_{m=0}^{j}\binom{n+1}{j-m}\frac{r^m\sin[mt_1+t_2+(n+1+\alpha)\theta]}{(1+r^2-2r\cos\theta)^{\frac{m+1}{2}}}. \tag{2.36}$$

By (2.32), (2.33), (2.35), (2.36), (2.24), and (2.25), we obtain (2.30) and (2.31).

In (2.30) and (2.31), set $n\to\infty$, then we easily obtain the following conclusion.

Theorem 2.4: If $r<1$, $0\le\theta\le 2\pi$, then

$$\sum_{k=\alpha}^{\infty}D_p(k,\alpha)r^k\cos k\theta = \sum_{j=0}^{p} j!S(p,j,\alpha)\psi_2^{(1)}(\alpha,r,\theta,n,j), \tag{2.37}$$

$$\sum_{k=\alpha}^{\infty}D_p(k,\alpha)r^k\sin k\theta = \sum_{j=0}^{p} j!S(p,j,\alpha)\psi_2^{(2)}(\alpha,r,\theta,n,j), \tag{2.38}$$

where

$$\psi_2^{(1)}(\alpha,r,\theta,n,j) = r^{j+\alpha}\left[T_j\frac{\cos\alpha\theta - r\cos(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} - \frac{\sin\theta\sin\alpha\theta + r\sin\theta\sin(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right],$$

$$\psi_2^{(2)}(\alpha,r,\theta,n,j) = r^{j+\alpha}\left[T_j\frac{\sin\alpha\theta - r\sin(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+2}{2}}} + \frac{\sin\theta\cos\alpha\theta - r\sin\theta\cos(1-\alpha)\theta}{(1+r^2-2r\cos\theta)^{\frac{j+3}{2}}}\sum_{l=0}^{j-1}T_lT_1^{j-1-l}\right].$$

These imply that the rank of (2.37) and (2.38) is 3.

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