# ELEMENTARY PROBLEMS AND SOLUTIONS

*Edited by*

**Russ Euler and Jawad Sadek**

*Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.*

*If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.*

*Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by April 15, 2003. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".*

## BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

## PROBLEMS PROPOSED IN THIS ISSUE

**B-946** ***Proposed by Mario Catalani, University of Torino, Torino, Italy***

Find the smallest positive integer $k$ such that the following series converge and find the value of the sums:

**1.** $\displaystyle\sum_{i=1}^{\infty} \frac{i^2 F_i L_i}{k^i}$  **2.** $\displaystyle\sum_{i=1}^{\infty} \frac{i F_i^2}{k^i}$

**B-947** ***Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA***

***(a)*** Find a nonsquare polynomial $f(x,y,z)$ with integer coefficients such that $f(F_n, F_{n+1}, F_{n+2})$ is a perfect square for all $n$.

***(b)*** Find a nonsquare polynomial $g(x,y)$ with integer coefficients such that $g(F_n, F_{n+1})$ is a perfect square for all $n$.

**B-948** ***Proposed by José Luis Díaz-Barrero & Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain***

Let $\ell$ be a positive integer greater than or equal to 2. Show that, for $x > 0$,

$$\log_{F_{\ell+1}F_{\ell+2}\cdots F_{\ell+n}} x^{n^2} \le \sum_{k=1}^{n} \log_{F_{\ell+k}} x.$$

**B-949** ***Proposed by N. Gauthier, Royal Military College of Canada***

For $l$ and $n$ positive integers, find closed form expressions for the following sums,

$$S_1 \equiv \sum_{k=1}^{n} 3^{n-k} F^3_{3^k \cdot 2l} \quad \text{and} \quad S_2 \equiv \sum_{k=1}^{n} 3^{n-k} L^3_{3^k \cdot (2l+1)}.$$

**B-950** ***Proposed by Paul S. Bruckman, Berkeley, CA***

For all primes $p > 2$, prove that

$$\sum_{k=1}^{p-1} \frac{F_k}{k} \equiv 0 \pmod{p},$$

where $\frac{1}{k}$ represents the residue $k^{-1} \pmod{p}$.

## SOLUTIONS

### An Inequality and an Equality Case

**B-930** ***Proposed by José Luis Díaz & Juan José Egozcue, Terrassa, Spain (Vol. 40, no. 1, February 2002)***

Let $n \ge 0$ be a nonnegative integer. Prove that $F_n^{L_n} L_n^{F_n} \le (F_{n+1}^{F_{n+1}})^2$. When does equality occur?

***Solution by H.-J. Seiffert, Berlin, Germany***

We shall prove that, for all nonnegative integers $n$,

$$F_n^{L_n} L_n^{F_n} \le F_{2n}^{F_{n+1}} \le F_{n+1}^{2F_{n+1}}, \tag{1}$$

with equality on the left-hand side only when $n = 0$ or $n = 1$, and on the right-hand side only when $n = 1$.

If $x$ and $y$ are distinct positive real numbers, then, by the weighted Arithmetic-Geometric Mean Inequality,

$$x^y y^x < \left(\frac{2xy}{x+y}\right)^{x+y}.$$

Since $\frac{2xy}{x+y} < \sqrt{xy} < \frac{x+y}{2}$, we have

$$x^y y^x < (xy)^{(x+y)/2} < \left(\frac{x+y}{2}\right)^{x+y}. \tag{2}$$

The cases $n = 0$ and $n = 1$ can be treated directly. If $n \ge 2$, then $0 < F_n < L_n$ and (2) gives

$$F_n^{L_n} L_n^{F_n} < (F_n L_n)^{(F_n + L_n)/2} < \left(\frac{F_n + L_n}{2}\right)^{F_n + L_n}.$$

From ($I_7$) and ($I_8$) of [1], we know that $F_n L_n = F_{2n}$ and $L_n = F_{n-1} + F_{n+1}$. The latter identity gives $F_n + L_n = 2F_{n+1}$ so that (1), including the conditions for equality, is proved.

**Reference**

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

***Also solved by Paul Bruckman, Charles Cook, L.A.G. Dresel, Ovidiu Furdui, Walther Janous, Harris Kwong, Toufik Mansour (partial solution), and the proposer.***

## A Relatively Prime Couple

**B-931** *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 40, no. 1, February 2002)*

Prove that $\gcd(L_n, F_{n+1}) = 1$ for all $n \geq 0$.

***Solution by Charles K. Cook, University of South Carolina at Sumter, Sumter, SC***

First, note that $L_n = F_{n+1} + F_{n-1}$ and $F_n = F_{n+1} - F_{n-1}$. Let $d = (F_{n+1}, L_n)$. Then $d \mid F_{n+1}$ and $d \mid L_n = F_{n+1} + F_{n-1}$. It follows that $d \mid F_{n-1}$. Thus, $d \mid F_{n+1} - F_{n-1}$ and $F \mid F_n$. But $(F_n, F_{n+1}) = 1$. Hence, $d = 1$.

***Also solved by Paul Bruckman, L.A.G. Dresel, Pentti Haukkanen, John Jaroma, Walther Janous, Harris Kwong, Toufik Mansour, Maitland Rose, H.-J. Seiffert, and the proposer.***

## A Strict Inequality and a Serious Series

**B-932** *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 40, no. 1, February 2002)*

Prove that

**A)** $\dfrac{F_2F_4\ldots F_{2n}}{F_1F_3\ldots F_{2n+1}} < \dfrac{1}{\sqrt{F_{2n+1}}}$ for all $n \geq 1$ and **B)** $\displaystyle\sum_{k=1}^{\infty} \frac{F_2F_4\ldots F_{2k}}{F_1F_3\ldots F_{2k+1}}$ converges.

***Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY***

The inequality in **A)** can be established by means of induction. To complete the inductive step, it suffices to prove that

$$\frac{1}{\sqrt{F_{2n+1}}} \cdot \frac{F_{2n+2}}{F_{2n+3}} < \frac{1}{\sqrt{F_{2n+3}}}$$

or, equivalently,

$$F_{2n+2}^2 < F_{2n+1}F_{2n+3}.$$

Binet's formulas yield

$$5F_{2n+2}^2 = (\alpha^{2n+2} - \beta^{2n+2})^2 = \alpha^{4n+4} - 2(\alpha\beta)^{2n+2} + \beta^{4n+4} = L_{4n+4} - 2$$

and, in a similar manner,

$$5F_{2n+1}F_{2n+3} = \alpha^{4n+4} - 2(\alpha\beta)^{2n+1}(\alpha^2 + \beta^2) + \beta^{4n+4} = L_{4n+4} + 2L_3,$$

thereby completing the induction.

To prove **B)**, we use a comparison test. Because of **A)**, it remains to show that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{F_{2k+1}}}$$

converges. We now apply a ratio test to complete the proof:

$$\lim_{k\to\infty} \frac{1}{\sqrt{F_{2k+3}}} \Bigg/ \frac{1}{\sqrt{F_{2k+1}}} = \lim_{k\to\infty} \sqrt{\frac{F_{2k+1}}{F_{2k+3}}} = \frac{1}{\alpha} < 1.$$

*H.-J. Seiffert proved the sharper inequality*

$$\frac{F_2F_4\dots F_{2n}}{F_1F_3\dots F_{2n+1}} \le \frac{1}{\sqrt{2F_{2n+1}}}$$

*with equality occurring only when* $n = 1$.

***Also solved by Paul Bruckman, Charles Cook, L.A.G. Dresel, José Luis Díaz & Juan José Egozcue (jointly), Pentti Haukkanen, Walther Janous, Toufik Mansour, H.-J. Seiffert, and the proposer.***

## A Special Case of a More General Inequality

**B-933** ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 40, no. 1, February 2002)***

Prove that $F_n^{F_{n+1}} > F_{n+1}^{F_n}$ for all $n \ge 4$.

***Solution I by Pantelimon Stănică, Montgomery, AL***

We prove the inequality by induction. For $n = 4$, we need $3^5 > 5^3$, which is obviously true. Assuming the inequality true for $n$, we prove it for $n + 1$. Thus,

$$F_{n+1}^{F_{n+2}} = F_{n+1}^{F_{n+1}}F_{n+1}^{F_n} > F_{n+1}^{F_{n+1}}F_n^{F_{n+1}} = (F_{n+1}F_n)^{F_{n+1}} > F_{n+2}^{F_{n+1}},$$

using induction and the fact that $F_nF_{n+1} > F_n + F_{n+1} = F_{n+2}$.

*Further Comment:* In fact, a much more general inequality is true (see, e.g., F. Qi & L. Debnath, "Inequalities of Power-Exponential Functions," *J. Ineq. Pure Appl. Math.* **1.2** [2000]:art. 15):

If $e < x < y$, then $x^y > y^x$;

If $x < y < e$, then $x^y < y^x$.

The first inequality, taking $x = F_n$, $y = F_{n+1}$, $n \ge 4$, implies **B-933**.

***Solution II by Paul S. Bruckman, Sacramento, CA***

For brevity, write $a = F_n$, $b = F_{n+1}$.

Consider the analytic function $F(x) = x / \ln x$, $x > 1$. Note that $F'(x) = 1/\ln x - 1/\ln^2 x$; thus $F'(x) = 0$ iff $x = e$. Also, $F''(x) = -1/(x\ln^2 x) + 2/(x\ln^3 x)$. Since $F''(e) = 1/e > 0$, then $F$ attains a relative minimum at $x = e$; in fact, $F(e) = e$. Moreover, $F'(x) > 0$ if $x > e$, i.e., $F$ is increasing for all $x > e$.

In particular, if $n \ge 4$, we have $b \ge 5$, $b > a \ge 3 > e$, so $b/\ln b > a/\ln a$. Equivalently, $b\ln a > a\ln b$, which implies $a^b > b^a$. Q.E.D.

Incidentally, note that if $n < 4$, $a < e$, then the indicated inequality is invalid. For $n = 1, 2, 3$, respectively, we find that $1 = 1^1 = 1^1$, $1 = 1^2 < 2^1 = 2$, and $8 = 2^3 < 3^2 = 9$.

*Most solvers used variations of Solution II. H.-J. Seiffert improved the inequality by showing*

$$F_n^{F_{n+1}} > \left(\frac{15}{4e}\right)^2 F_{n+1}^{F_n}.$$

***Also solved by Charles Cook, José Luis Díaz-Barrero & Juan José Egozcue (jointly), L.A.G. Dresel, Walther Janous, Toufik Mansour, H.-J. Seiffert, and the proposer.***

## A Trigonometric Fibonacci Equality

**B-934** ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 40, no. 1, February 2002)***

Prove that

$$2\sum_{n=1}^{m}\sin^2\left(\frac{\pi}{2}\frac{F_{m+1}}{F_mF_nF_{n+1}}\right)\sin\left(\pi\frac{F_nF_{m+1}}{F_mF_{n+1}}\right)=\sum_{n=1}^{m}(-1)^n\sin\left(\pi\frac{F_{m+1}}{F_mF_nF_{n+1}}\right)\cos\left(\pi\frac{F_nF_{m+1}}{F_{n+1}F_m}\right),$$

where $m$ is a positive integer.

***Solution by L. A. G. Dresel, Reading, England***

For any given $m$, let $A_n=\pi F_{m+1}/(F_mF_nF_{n+1})$ and $B_n=\pi F_nF_{m+1}/(F_mF_{n+1})$. Using the trigonometric identities $2\sin^2(A/2)=1-\cos A$, $\sin(B+A)=\cos A\sin B+\cos B\sin A$, and $\sin(B-A)=\cos A\sin B-\cos B\sin A$, the proposition to be proved transforms to

$$\Sigma\sin B_n=\Sigma\sin\{B_n+(-1)^nA_n\},$$

the summations being from $n=1$ to $m$. Now

$$\{B_n+(-1)^nA_n\}=\{\pi F_{m+1}/(F_mF_nF_{n+1})\}\{(F_n)^2+(-1)^n\},$$

and using identity (29) of [1], $(F_n)^2+(-1)^n=F_{n+1}F_{n-1}$, we obtain

$$\{B_n+(-1)^nA_n\}=\pi F_{n-1}F_{m+1}/(F_mF_n)=B_{n-1}.$$

Thus, it remains to prove $\Sigma(\sin B_n-\sin B_{n-1})=0$, which reduces to $(\sin B_m-\sin B_0)=0$. But $B_0=0$ and $B_m=\pi$, so that $\sin B_m=\sin B_0=0$.

### Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Chichester: Ellis Horwood Ltd., 1989.

***Also solved by Paul Bruckman, Ovidiu Furdui, Walther Janous, H.-J. Seiffert, and the proposer.***