ON THE PRODUCT OF LINE-SEQUENCES

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For consistency, we adopt the same notations and formats developed in our previous work on line-sequences, see [2].

A line-sequence is expressed as

$$\bigcup_{u_0,\, u_1}(c, b): \ldots u_{-3}, u_{-2}, u_{-1}, [u_0, u_1], u_2, u_3, u_4, \ldots, \tag{1}$$

where u_n, $n \in Z$, denotes the n^{th} term, the generating pair is given by $[u_0, u_1]$, and the recurrence relation is

$$cu_n + bu_{n+1} = u_{n+2}, \tag{2}$$

where $c, b \in R$ are not zero. Since (2) is valid for any value of n, we also have

$$cu_{n+1} + bu_{n+2} = u_{n+3}.$$

From these two relations, we find

$$b = (u_n u_{n+3} - u_{n+1}u_{n+2}) / (u_n u_{n+2} - (u_{n+1})^2), \tag{3}$$

$$c = ((u_{n+2})^2 - u_{n+1}u_{n+3}) / (u_n u_{n+1} - (u_{n+1})^2). \tag{4}$$

The product (see, e.g., [1], [4], [5]), abbreviated as "product" here, of two line-sequences does not necessarily satisfy a recurrence relation. We will give some conditions under which it does.

A generalized Fibonacci line-sequence is given by

$$\bigcup_{0,1}(c, b): \ldots [0, 1], b, c + b^2, \ldots, \tag{5}$$

and a generalized Lucas line-sequence is given by

$$\bigcup_{2,b}(c, b): \ldots [2, b], 2c + b^2, 3cb + b^3, \ldots \tag{6}$$

see (4.3) and (4.12) in [2]. Let

$$\bigcup_{0,b}(y, x) = \bigcup_{0,1}(c, b)\bigcup_{2,b}(c, b). \tag{7}$$

Substituting (5) and (6) into (7) and multiplying corresponding terms produces

$$\bigcup_{0,b}(y, x): \ldots [0, b], 2cb + b^3, 3c^2b + 4cb^3 + b^5, \ldots. \tag{8}$$

Putting $n = 0$ in (3) and (4) and applying to (8), we obtain

$$x = 2c + b^2, \ \ y = -c^2. \tag{9}$$

So (7) becomes

$$\bigcup_{0,b}(-c^2, 2c + b^2) = \bigcup_{0,1}(c, b)\bigcup_{2,b}(c, b). \tag{10}$$

Let

$$\bigcup_{-b,0}(y,x)=\bigcup_{1,0}(c,b)\bigcup_{-b,2c}(c,b). \tag{11}$$

Following the same procedure, we find

$$\bigcup_{-b,0}(-c^2,2c+b^2)=\bigcup_{1,0}(c,b)\bigcup_{-b,2c}(c,b). \tag{12}$$

From (10) and (12), we have the following pair:

$$\bigcup_{1,0}(-c^2,2c+b^2)=-(1/b)\bigcup_{1,0}(c,b)\bigcup_{-b,2c}(c,b), \tag{13}$$

$$\bigcup_{0,1}(-c^2,2c+b^2)=(1/b)\bigcup_{0,1}(c,b)\bigcup_{2,b}(c,b). \tag{14}$$

So we obtain the formula:

$$\begin{aligned}\bigcup_{i,j}(-c^2,2c+b^2)&=i\bigcup_{1,0}(-c^2,2c+b^2)+j\bigcup_{0,1}(-c^2,2c+b^2)\\&=(1/b)\left[-i\bigcup_{1,0}(c,b)\bigcup_{-b,2c}(c,b)+j\bigcup_{0,1}(c,b)\bigcup_{2,b}(c,b)\right].\end{aligned} \tag{15}$$

Example: Let $c=b=1$ in (15) and put $M_{i,j}=\bigcup_{i,j}(-1,3)$ and $F_{i,j}=\bigcup_{i,j}(1,1)$, then

$$M_{i,j}=-iF_{1,0}F_{-1,2}+jF_{0,1}F_{2,1}, \tag{16}$$

where M denotes Morgan-Voyce numbers, see (1) in [3].

Let $m_{i,j;n}$ and $f_{i,j;n}$ be the n^{th} term of $M_{i,j}$ and $F_{i,j}$, respectively. Then

$$m_{i,j;n}=-if_{1,0;n}f_{-1,2;n}+jf_{0,1;n}f_{2,1;n}=-if_{n-1}l_{n-1}+jf_nl_n, \tag{17}$$

where f_n and l_n denote the n^{th} Fibonacci and the n^{th} Lucas numbers, respectively. In particular,

$$m_{1,0;n}=-f_{n-1}l_{n-1}=-f_{2n-2}, \tag{18}$$

$$m_{0,1;n}=f_nl_n=f_{2n}. \tag{19}$$

Since the generating function of $M_{i,j}$ is $(j-it)/(1-3t+t^2)$, we have

$$t/(1-3t+t^2)=\sum_{n\ge1}f_{2n-2}t^{n-1}, \tag{20}$$

and

$$1/(1-3t+t^2)=\sum_{n\ge1}f_{2n}t^{n-1}. \tag{21}$$

For $M_{1,1}$,

$$(1-t)/(1-3t+t^2)=\sum_{n\ge1}f_{2n-1}t^{n-1}, \tag{22}$$

and for $M_{-1,1}$,

$$(1+t)/(1-3t+t^2)=\sum_{n\ge1}l_{2n-1}t^{n-1}. \tag{23}$$

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