A NOTE ON A CLASS OF COMPUTATIONAL FORMULAS INVOLVING THE MULTIPLE SUM OF RECURRENCE SEQUENCES

Zhizheng Zhang
Department of Mathematics, Nanjing University,
Nanjing, Jiangsu 210093, P.R. China

Xin Wang
Department of Basics, Dalian Naval Academy, Dalian, Liaoning 116018, P.R. China
(Submitted October 1999-Final Revision April 2002)

1. INTRODUCTION

The second-order linear recurrence sequence $U=\{U_n\}$, $n=0,1,2,\ldots$, is defined by integers a, b, U_0, U_1 and by the recursion $U_{n+2}=bU_{n+1}+aU_n$ for $n\geq 0$. We suppose that $ab\neq 0$ and not both U_0 and U_1 are zero. If α and β denote the roots of the characteristic polynomial x^2-bx-a of the sequence U, then we have the Binet formula (see [1]):

$$U_n=\frac{A\alpha^n-B\beta^n}{\alpha-\beta},$$

where $A=U_1-U_0\beta$ and $B=U_1-U_0\alpha$. The generating function is

$$\sum_{n=0}^{\infty}U_nx^n=\frac{U_0+(U_1-U_0b)x}{1-bx-ax^2}.$$

If $U_0=0$, $U_1=1$, then the sequence $\mathcal{F}=\{U_n\}$ is called the generalized Fibonacci sequence, and $\mathcal{F}_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$.

In order to express our results, we denote by $\sigma_{i,j}(n,k)$ (i, j, and k are nonnegative integers) the summation of all products of choosing j elements from $n+2k-1$, $n+2k-2$, ..., $n+2k-i+1$ but not containing any two consecutive elements. We note that $\sigma_{i,j}(n,k)=0$ if $j<0$ or $j>[\frac{i}{2}]$, $\sigma_{i,0}(n,k)=1$ $(i\geq 0)$, $\sigma_{i,1}(n,k)=\frac{1}{2}(i-1)(2n+4k-i)$ $(i\geq 1)$. For example, when $i=6$, we have

$$\begin{aligned}
&\sigma_{6,0}(n,k)=1,\\
&\sigma_{6,1}(n,k)=(n+2k-1)+(n+2k-2)+(n+2k-3)+(n+2k-4)+(n+2k-5),\\
&\sigma_{6,2}(n,k)=(n+2k-1)(n+2k-3)+(n+2k-1)(n+2k-4)+(n+2k-1)(n+2k-5)\\
&\qquad+(n+2k-2)(n+2k-4)+(n+2k-2)(n+2k-5)+(n+2k-3)(n+2k-5),\\
&\sigma_{6,3}(n,k)=(n+2k-1)(n+2k-3)(n+2k-5).
\end{aligned}$$

It is easy to prove that

$$(n+2k-1)\sigma_{2k-2,k-1}(n,k-1)=\sigma_{2k,k}(n,k)\quad(k\geq 1)$$

and

$$(n+2k-1)\sigma_{k+i-2,i-1}(n,k-1)+\sigma_{k+i-1,i}(n+1,k-1)=\sigma_{k+i,i}(n,k)\quad(1\leq i\leq k,\ k\geq 2).$$

Recently, W. Zhang [2] obtained the following result: Let $U=(U_n\}$ be defined as above. If $U_0=0$, then for any positive integer $k\geq 2$, we have

$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}U_{a_2}\ldots U_{a_k} = \frac{U_1^{k-1}}{(b^2+4a)^{k-1}(k-1)!}[g_{k-1}(n)U_{n-k+1}+h_{k-1}(n)U_{n-k}],$$

where the summation is taken over all n-tuples with positive integer coordinates $(a_1, a_2, \ldots, a_k)$ such that $a_1+a_2+\cdots+a_k=n$, and he pointed out that $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials of degree $k-1$, their coefficients depending only on a, b, and k.

In this paper, we obtain

$$g_{k-1}(n) = \sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1, k-1)\mathcal{F}_{k-i} \quad (k\geq 1)$$

and

$$h_{k-1}(n) = a\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1, k-1)\mathcal{F}_{k-i-1} \quad (k\geq 1),$$

where $\langle n\rangle_k = n(n+1)\cdots(n+k-1)$ with $\langle n\rangle_0 = 1$. We also give the congruence relation

$$g_{k-1}(n)U_{n-k+1}+h_{k-1}(n)U_{n-k} \equiv 0 \pmod{(k-1)!(b^2+4a)^{k-1}} \quad (k\geq 1),$$

which generalizes the results presented in [2].

2. THE RESULTS AND THEIR PROOFS

In this section, with $U_0 = 0$, let

$$G_k(x) = \left(\frac{U_1}{1-bx-ax^2}\right)^k = \sum_{n=0}^{\infty} U_n^{(k)}x^{n-1}.$$

Then

$$\sum_{a_1+a_2+\cdots+a_m=n} U_{a_1}^{(k_1)}U_{a_2}^{(k_2)}\cdots U_{a_m}^{(k_m)} = U_{n-m+1}^{(k_1+k_2+\cdots+k_m)}.$$

Taking $k_1 = k_2 = \cdots = k_m = 1$, we have

Lemma 1: $\sum_{a_1+a_2+\cdots+a_m=n} U_{a_1}U_{a_2}\cdots U_{a_m} = U_{n-m+1}^{(m)}.$

Theorem 1: $U_n^{(k+1)} = \dfrac{U_1}{k(b^2+4a)}\{nbU_{n+1}^{(k)}+2a(n+2k-1)U_n^{(k)}\}\quad (k\geq 1).$

Proof:

$$\frac{d}{dx}(G_k(x)(b+2ax)^k) = G_k'(x)(b+2ax)^k + G_k(x)k(b+2ax)^{k-1}2a$$

and

$$\frac{d}{dx}(G_k(x)(b+2ax)^k) = \frac{d}{dx}\left(\frac{U_1(b+2ax)}{1-bx-ax^2}\right)^k$$

$$= kU_1\left(\frac{U_1(b+2ax)}{1-bx-ax^2}\right)^{k-1}\frac{2a(1-bx-ax^2)+(b+2ax)^2}{(1-bx-ax^2)^2}$$

$$= k(b+2ax)^{k-1}U_1\left(\frac{U_1}{1-bx-ax^2}\right)^{k-1}\frac{2a^2x^2+2abx+b^2+2a}{(1-bx-ax^2)^2}$$

$$= k(b+2ax)^{k-1}U_1\left(\frac{U_1}{1-bx-ax^2}\right)^{k-1}\frac{-2a(1-bx-ax^2)+b^2+4a}{(1-bx-ax^2)^2}.$$

Hence,

$$G_k'(x)(b+2ax)^k+G_k(x)k(b+2ax)^{k-1}2a$$

$$= k(b+2ax)^{k-1}U_1\left(\frac{U_1}{1-bx-ax^2}\right)^{k-1}\frac{-2a(1-bx-ax^2)+b^2+4a}{(1-bx-ax^2)^2}.$$

Therefore,

$$G_k'(x)U_1(b+2ax)+2akU_1G_k(x)=-2akU_1G_k(x)+(b^2+4a)kG_{k+1}(x).$$

This concludes the proof of Theorem 1. □

Theorem 2: $U_n^{(k+1)}=\dfrac{U_1^k}{k!(b^2+4a)^k}\sum_{i=0}^{k}(2a)^i b^{k-i}\langle n\rangle_{k-i}\sigma_{k+i,i}(k)U_{n+k-i}\quad (k\geq 0).$

Proof: This theorem can be proved by induction. When $k=0$, the theorem is trivial. When $k=1$, the theorem is true by applying Theorem 1. Assume the theorem is true for a positive integer $k-1$, then

$$U_n^{(k+1)}=\frac{U_1}{k(b^2+4a)}\{nbU_{n+1}^{(k)}+2a(n+2k-1)U_n^{(k)}\}$$

$$=\frac{U_1}{k(b^2+4a)}\left\{nb\frac{U_1^{k-1}}{(k-1)!(b^2+4a)^{k-1}}\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n+1\rangle_{k-i-1,i}\sigma_{k+i-1,i}(n+1,k-1)U_{n+k-i}\right.$$

$$\left.+2a(n+2k-1)\frac{U_1^{k-1}}{(k-1)!(b^2+4a)^{k-1}}\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n\rangle_{k-i-1}\sigma_{k+i-1,i}(n,k-1)U_{n+k-i-1}\right\}$$

$$=\frac{U_1^k}{k!(b^2+4a)^k}\left\{\sum_{i=0}^{k-1}(2a)^i b^{k-n}\langle n+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n+1,k-1)U_{n+k-i}\right.$$

$$\left.+\sum_{i=0}^{k-1}(2a)^{i+1}b^{k-i-1}\langle n\rangle_{k-i-1}(n+2k-1)\sigma_{k+i-1,i}(n,k-1)U_{n+k-i-1}\right\}$$

$$=\frac{U_1^k}{k!(b^2+4a)^k}\left\{\sum_{i=0}^{k-1}(2a)^i b^{k-i}\langle n\rangle_{k-i}\sigma_{k+i-1,i}(n+1,k-1)U_{n+k-i}\right.$$

$$\left.+\sum_{i=0}^{k}(2a)^i b^{k-i}\langle n\rangle_{k-i}(n+2k-1)\sigma_{k+i-2,i-1}(n,k-1)U_{n+k-i}\right\}$$

$$=\frac{U_1^k}{k!(b^2+4a)^k}\left\{b^k\langle n\rangle_k\sigma_{k-1,0}(n+1,k-1)U_{n+k}+\sum_{i=1}^{k-1}(2a)^i b^{k-i}\langle n\rangle_{k-i}U_{n+k-i}[\sigma_{k+i-1,i}(n,k-1)\right.$$

$$\left.+(n+2k-1)\sigma_{k+i-2,i-1}(n,k-1)]+(2a)^k(n+2k-1)\sigma_{2k-2,k-1}(n,k-1)U_n\right\}$$

$$=\frac{U_1^k}{k!(b^2+4a)^k}\left\{b^k\langle n\rangle_k\sigma_{k,0}(n+1,k)U_{n+k}+\sum_{i=1}^{k-1}(2a)^i b^{k-i}\langle n\rangle_{k-i}U_{n+k-i}\sigma_{k+i,i}(n,k)+(2a)^k\sigma_{2k,k}(n,k)U_n\right\}$$

$$=\frac{U_1^k}{k!(b^2+4a)^k}\sum_{i=0}^{k}(2a)^i b^{k-i}\langle n\rangle_{k-i}\sigma_{k+i,i}(n,k)U_{n+k-i}.$$

That is, the theorem is also true for k. This proves the Theorem 2. □

Lemma 2: $U_{m+k}=\mathscr{F}_{k+1}U_m+a\mathscr{F}_k U_{m-1}$ $(k\ge 0, m\ge 1)$.

Proof: Use Binet's formula. □

Theorem 3: $U_n^{(k+1)}=\dfrac{U_1^k}{k!(b^2+4a)^k}\sum_{i=0}^{k}(2a)^i b^{k-i}\langle n\rangle_{k-i}\sigma_{k+i,i}(n,k)(\mathscr{F}_{k-i+1}U_n+a\mathscr{F}_{k-i}U_{n-1})$ $(k\ge 0)$.

Proof: Use Theorem 2 and Lemma 2. □

Theorem 4: $\sum_{a_1+a_2+\cdots+a_k=n}U_{a_1}U_{a_2}\dots U_{a_k}$

$$=\frac{U_1^{k-1}}{(b^2+4a)^{k-1}(k-1)!}\left\{\left[\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathscr{F}_{k-i}\right]U_{n-k+1}\right.$$
$$\left.+a\left[\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathscr{F}_{k-i-1}\right]U_{n-k}\right\}\quad(k\ge 1).$$

Proof: Noting Lemma 1 and Theorem 3, we have

$$\sum_{a_1+a_2+\cdots+a_k=n}U_{a_1}U_{a_2}\dots U_{a_k}=U_{n-k+1}^{(k)}$$
$$=\frac{U_1^{k-1}}{(k-1)!(b^2+4a)^{k-1}}\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-1-i}\sigma_{k-1+i,i}(n-k+1,k-1)$$
$$\times(\mathscr{F}_{k-i}U_{n-k+1}+a\mathscr{F}_{k-i-1}U_{n-k})$$
$$=\frac{U_1^{k-1}}{(b^2+4a)^{k-1}(k-1)!}\left\{\left[\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathscr{F}_{k-i}\right]U_{n-k+1}\right.$$
$$\left.+a\left[\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathscr{F}_{k-i-1}\right]U_{n-k}\right\}.\ \square$$

From this theorem, we can get the expression of $g_{k-1}(n)$ and $h_{k-1}(n)$, namely,

$$g_{k-1}(n)=\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathscr{F}_{k-i}\quad(k\ge 1)$$

and

$$h_{k-1}(n)=a\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathscr{F}_{k-i-1}\quad(k\ge 1)..$$

Theorem 5: $g_{k-1}(n)U_{n-k+1}+h_{k-1}(n)U_{n-k}\equiv 0\ (\mathrm{mod}\,(k-1)!(b^2+4a)^{k-1})$ $(k\ge 1)$.

This result is a generalization of Corollary 2 of [2]. When $U_1 = a = b = 1$ and $k = 1, 2, 3$, respectively, this result becomes (i)-(iii) of Corollary 2 of [2].

ACKNOWLEDGMENT

The authors wish to thank the anonymous referee for his/her patience and suggestions that have significantly improved the presentation of this paper.

This work was supported by the National Natural Science Foundation of China.

REFERENCES

1. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.2** (1965):161-76.
2. Wenpeng Zhang. "Some Identities Involving the Fibonacci Numbers." *The Fibonacci Quarterly* **35.3** (1997):225-29.

AMS Classification Numbers: 11B37, 11B39, 06B10

❖❖❖

THE PASSING OF THREE FIBONACCI ASSOCIATION FRIENDS

We were all deeply saddened to learn of the recent deaths of **Joe Arkin, Daniel Fielder** and **Peter Kiss**. These three long-time members of the Fibonacci Association will be greatly missed.