ADVANCED PROBLEMS AND SOLUTIONS

Edited by V. E. HOGGATT, JR., San Jose State College, San Jose, Calif.

Send all communications concerning Advanced Problems and Solutions to Raymond Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within three months after publication of the problems.

NOTICE: PLEASE SEND ALL SOLUTIONS AND NEW PROPOSALS TO PROFESSOR RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA.

H-103 **Proposed by David Zeitlin, Minneapolis, Minnesota.**

Show that

$$8\sum_{k=0}^{n} F_{3k+1}F_{3k+2}F_{6k+3} = F^4_{3n+3}\ .$$

H-104 **Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.**

Show

$$\frac{L_mX}{1 - 5F_mX + (-1)^{m+1}5X^2} = \sum_{k=0}^{\infty} 5^k(F_{2mk} + XL_{(2k+1)m})X^{2k}\ ,$$

where L_m and F_m are the m^{th} Lucas and Fibonacci numbers, respectively.

H-105 **Proposed by Edgar Karst, Norman, Oklahoma, and S. O. Rorem, Davenport, Iowa.**

Show for all positive integral n and primes $p > 2$ that

$$(n + 1)^p - n^p = 6N + 1\ ,$$

where N is a positive integer. Generalize.

H-106 **Proposed by L. Carlitz, Duke University, Durham, N. Carolina.**

Show that

a)
$$\sum_{k=0}^{n} \binom{n}{k}^2 L_{2k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} L_{n-k}$$

b)
$$\sum_{k=0}^{n} \binom{n}{k}^2 F_{2k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} F_{n-k}$$

H-107 **Proposed by Vladimir Ivanoff, San Carlos, California.**

Show that

$$\begin{vmatrix} F_{p+2n} & F_{p+n} & F_p \\ F_{q+2n} & F_{q+n} & F_q \\ F_{r+2n} & F_{r+n} & F_r \end{vmatrix} = 0$$

for all integers p, q, r, and n.

H-108 **Proposed by H. E. Huntley, Hutton, Somerset, U.K.**

Find the sides of a tetrahedron, the faces of which are all scalene triangles similar to each other, and having sides of integral lengths.

H-109 **Proposed by George Ledin, Jr., San Francisco, California.**

Solve

$$X^2 + Y^2 + 1 = 3XY$$

for all integral solutions and consequently derive the identity:

$$F_{6k+7}^2 + F_{6k+5}^2 + 1 = 3F_{6k+7}F_{6k+5} .$$

H-110 **Proposed by George Ledin, Jr., San Francisco, California**

Evaluate the double sum

$$S_n = \sum_{m=1}^{n} \sum_{k=1}^{\infty} F_{\left[\frac{m}{k}\right]}$$

where [x] is the greatest integer in x.

H-111 Proposed by John L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania.

Show that

$$L_n = \prod_{k=1}^{\left[\frac{n}{2}\right]} \left\{1 + 4\cos^2 \frac{2k-1}{n}\left(\frac{\pi}{2}\right)\right\} \quad \text{for } n \geq 1 .$$

H-112 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that, for $n \geq 1$,

a) $L_{n+1}^5 - L_n^5 - L_{n-1}^5 = 5L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)$

b) $F_{n+1}^5 - F_n^5 - F_{n-1}^5 = 5F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)$

c) $L_{n+1}^7 - L_n^7 - L_{n-1}^7 = 7L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)^2$

d) $F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)^2$.

SOLUTIONS

NO SOLUTIONS RECEIVED

H-59 Proposed by D. W. Robinson, Brigham Young University, Provo, Utah.

Show that, if $m > 2$, then the period of the Fibonacci sequence 0, 1, 1, 2, 3, ⋯, F_n, ⋯ reduced modulo m is twice the least positive integer n such that $F_{n+1} \equiv (-1)^n F_{n-1}$ (mod m).

H-60 Proposed by Verner E. Hoggatt, San Jose State College, San Jose, Calif.

It is well known that if p_k is the least integer such that $F_{n+pk} = F_n$ mod 10^k, then $p_1 = 60$, $p_2 = 300$ and $p_k = 1.5 \times 10^k$ for $k \geq 3$. If Q(n, k) is the k^{th} digit of the n^{th} Fibonacci, then for fixed k, Q(n, k) is periodic, that is q_k is the least integer such that $Q(n + q_k, k) = Q(n, k)$ mod 10. Find an explicit expression for q_k.

H-62 Proposed by H. W. Gould, W. Virginia University, Morgantown, West Virginia (corrected).

Find all polynomials f(x) and g(x), of the form

$$f(x+1) = \sum_{j=0}^{r} a_j x^j, \quad a_j \text{ an integer}$$

$$g(x) = \sum_{j=0}^{s} b_j x^j, \quad b_j \text{ an integer}$$

such that

$$2\left\{x^2 f^3(x+1) - (x+1)^2 g^3(x)\right\} + 3\left\{x^2 f^2(x+1) - (x+1)^2 g^2(x)\right\}$$
$$+ (2x+1)\left\{x f(x+1) - (x+1) g(x)\right\} = 0 .$$

LIMIT OF LIMITS

H-61 Proposed by P. F. Byrd, San Jose State College, San Jose, Calif. (corrected)

Let

$$f_{n,k} = 0 \quad \text{for} \quad 0 \le n \le k-2, \quad f_{k-1,k} = 1 \quad \text{and}$$

$$f_{n,k} = \sum_{j=1}^{k} f_{n-j,k} \qquad \text{for} \quad n \ge k .$$

Show that

$$\frac{1}{2} < \frac{f_{n,k}}{f_{n+1,k}} < \frac{1}{2} + \frac{1}{2k} \quad \text{as } n \to \infty .$$

Hence

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{f_{n,k}}{f_{n+1,k}} = \frac{1}{2} .$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

The sequence $\{f_{n,k}\}_{n=0}^{\infty}$ obeys a recurrence whose auxiliary polynomial is

$$f(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 .$$

Let $r_{1,k}, r_{2,k}, \cdots, r_{k,k}$ be the k roots of $f(x) = 0$. The k initial conditions given determine constants $b_{1,k}, b_{2,k}, \cdots, b_{k,k}$ such that

$$f_{n,k} = \sum_{j=1}^{k} b_{j,k} r_{j,k}^{n} \quad .$$

Now Miles ["Generalized Fibonacci Numbers and Associated Matrices," Amer. Math. Monthly, Vol. 67, pp. 745-57] has shown that all but one of the roots $r_{j,k}$ lie within the unit circle, so that $|r_{j,k}| < 1$ $(1 \le j < k)$. Note that $f(1) = 1 - k < 0$, $f(2) = 1$, and since f is continuous, the remaining root $r_{k,k}$ must be a real number between 1 and 2. Then $b_{k,k} \neq 0$, because $\lim_{n\to\infty} r_{j,k}^{n} = 0$ $(1 \le j < k)$ while $\lim_{n\to\infty} f_{n,k} = \infty$. We also have

$$\lim_{n\to\infty} \frac{r_{j,k}^{n}}{r_{k,k}^{n}} = 0 \qquad (1 \le j < k) \ ,$$

so that

$$\lim_{n\to\infty} f_{n,k}/f_{n+1,k} = \lim_{n\to\infty} \left(\sum_{j=1}^{k} b_{j,k} r_{j,k}^{n}\right) \Big/ \left(\sum_{j=1}^{k} b_{j,k} r_{j,k}^{n+1}\right) = 1/r_{k,k} \ .$$

We have already shown $r_{k,k} < 2$. Now

$$\begin{aligned} &(2k)^k - (2k)^{k-1}(k+1) - (2k)^{k-2}(k+1)^2 - \cdots - (2k)(k+1)^{k-1} - (k+1)^k \\ &< (2k)^k - (2k)^{k-1}k - (2k)^{k-2}k - \cdots - (2k)k^{k-1} - k^k - k^k \\ &= 2^k k^k - k^k(2^{k-1} + 2^{k-2} + \cdots + 2 + 1 + 1) = 0 \ , \end{aligned}$$

and division by $(k+1)^k$ shows

$$f\left(\frac{2k}{k+1}\right) < 0 \quad .$$

Since

$$1 < \frac{2k}{k+1} < 2 \ ,$$

we have $2 > r_{k,k} > 2k/(k+1)$, and inversion gives the first result of the problem. The second result follows by taking limits as $k \to \infty$.

ODD ROW SUMS OF FIBONOMIAL COEFFICIENTS

H-63 Proposed by Stephen Jerbic, San Jose State College, San Jose, California.

Let

$$F(m,0) = 1 \text{ and } F(m,n) = \frac{F_m F_{m-1} \cdots F_{m-n+1}}{F_n F_{n-1} \cdots F_1} \qquad 0 < n \le m \ ,$$

be the Fibonomial coefficients, where F_n is the n^{th} Fibonacci number. Show

$$\sum_{n=0}^{2m-1} F(2m-1, n) = \prod_{i=0}^{m-1} L_{2i}, \quad m \ge 1 .$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Put
$$S_n = \sum_{r=0}^{n} F(n, r)$$

and

(1) $$f_n(x) = \sum_{r=0}^{n} (-1)^{r(r+1)/2} F(n, r) x^{n-r} .$$

Brennan ("Fibonacci Powers and Pascal's Triangle in a Matrix," Fibonacci Quarterly, Vol. 2, No. 2, pp. 93-103) has shown

$$f_n(x) = (-1)^{n-2}(x^2 - L_{n-1}x + (-1)^{n-1})f_{n-2}(-x)$$

where L_n is the n^{th} Lucas number. Setting $x = \sqrt{-1}$, $n = 2m + 1$, we find

$$f_{2m+1}(i) = iL_{2m} f_{2m-1}(-i) .$$

Using (1) this becomes

$$\sum_{r=0}^{m} F(2m+1, 2r) + i \sum_{r=0}^{m} F(2m+1, 2r+1)$$
$$= L_{2m} \sum_{r=0}^{m-1} F(2m-1, 2r+1) + iL_{2m} \sum_{r=0}^{m-1} F(2m-1, 2r) ,$$

and so equating real and imaginary parts, taking absolute values, and adding we get $S_{2m+1} = L_{2m} S_{2m-1}$ which, with $S_1 = 2 = L_0$, proves the proposition.

ONE OF MANY FORMS

H-64 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Show
$$F_{n+1} = \prod_{j=1}^{n} \left(1 - 2i \cos \frac{j\Pi}{n+1}\right) .$$

where F_n is the n^{th} Fibonacci number.

Solution by David Zeitlin, Minneapolis, Minnesota.

For a generalization, let W_0, W_1, $C \neq 0$, and $d \neq 0$ be arbitrary real numbers, and define $W_{n+2} = dW_{n+1} - cW_n$, $n = 0, 1, \cdots$, with $d^2 - 4c \neq 0$. We define $V_n \equiv W_n$, $n = 0, 1, \cdots$, when $W_0 = 1$ and $W_1 = d$; and set Z_n

$= W_n$, $n = 0, 1, \cdots$, when $W_0 = 0$ and $W_1 = 1$. In terms of Chebyshev polynomials of the first kind, $T_n(x)$, and of the second kind $U_n(x)$, it is readily verified that

$$(1) \qquad Z_{n+1} \equiv c^{n/2} U_n\left(\frac{d}{2\sqrt{c}}\right) \quad ; \qquad V_n \equiv 2c^{n/2} T_n\left(\frac{d}{2\sqrt{c}}\right)$$

Since

$$U_n(x) = 2^n \prod_{i=1}^{n} \left(x - \cos \frac{j\Pi}{n+1}\right) , \quad T_n(x) = 2^{n-1} \prod_{j=1}^{n} \left(x - \cos \frac{(2j-1)\Pi}{2n}\right) ,$$

we obtain from (1)

$$(2) \qquad Z_{n+1} = c^{n/2} \prod_{j=1}^{n} \left(\frac{d}{\sqrt{c}} - 2 \cos \frac{j\Pi}{n+1}\right) ,$$

$$(3) \qquad V_n = c^{n/2} \prod_{j=1}^{n} \left(\frac{d}{\sqrt{c}} - 2 \cos \frac{(2j-1)\Pi}{2n}\right)$$

If $d = 1$ and $c = -1$, then $Z_n = F_n$ and $V_n = L_n$. Since $-1 = i^2$, we obtain from (2) and (3), respectively,

$$(4) \qquad F_{n+1} = \prod_{j=1}^{n} \left(1 - 2i \cos \frac{j\Pi}{n+1}\right) ,$$

$$(5) \qquad L_n = \prod_{j=1}^{n} \left(1 - 2i \cos \frac{(2j-1)\Pi}{2n}\right) .$$

Also solved by F. D. Parker, John L. Brown, Jr., and the proposer.

FIBONACCI RELATED NUMBER

H-65 Proposed by J. Wlodarski, Porz-Westhoven, Federal Republic of Germany.

The units digit of a positive integer, M, is 9. Take the 9 and put it on the left of the remaining digits of M forming a new integer, N, such that $N = 9M$. Find the smallest M for which this is possible.

Solution by Robert H. Anglin, Danville, Va., and Murray Berg, Oakland, Calif.

$$M = 9 + \sum_{i=1}^{n} x_i 10^i$$

$$N = \sum_{i=1}^{n} x_i 10^{i-1} + 9 \cdot 10^n = 9M$$

$$10N = 90M = \sum_{i=1}^{n} x_i 10^i + 90 \cdot 10^n = M - 9 + 90 \cdot 10^n$$

$$89M = 9(10^{n+1} - 1)$$

$$M = \frac{9(10^{n+1} - 1)}{89} = \frac{89999\cdots 991}{89}$$

By performing the actual division the first zero occurs when the quotient is

M = 1011 23595 50561 79775 28089 88764 04494 38202 24719

Wlodarski notes

$$M = 10^{43} \div \left[10^{41} \sum_{m=1}^{\infty} \frac{F_m}{10^m} \right]$$

where $[x]$ is the greatest integer function in x.

Also solved by Marjorie Bicknell, James Desmond, A. B. Western, Jr., C.B.A. Peck, and the proposer.

A STIRLING NUMBER SOLUTION

H-66 **Proposed by Douglas Lind, University of Virginia, Charlottesville, Va., and Raymond E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.**

Let

$$\sum_{j=0}^{k} a_j y_{n+j} = 0$$

be a linear homogeneous recurrence relation with constant coefficients a_j. Let the roots of the auxiliary polynomial

$$\sum_{j=0}^{k} a_j x^j = 0$$

be $r_1, r_2, \cdots, r_m$ and each root r_i be of multiplicity m_i $(i = 1, 2, \cdots, m)$.

Jeske (Linear Recurrence Relations — Part I, Fibonacci Quarterly, Vol. No. 2, pp. 69-74) showed that

$$\sum_{n=0}^{\infty} y_n \frac{t^n}{n!} = \sum_{i=1}^{m} e^{r_i t} \sum_{j=0}^{m_i - 1} b_{ij} t^j .$$

He also stated that from this we may obtain

$$(\star) \qquad y_n = \sum_{i=1}^{m} r_i^n \sum_{j=0}^{m_i-1} b_{ij} n^j .$$

(i) Show that ($\star$) is in general incorrect, (ii) state under what conditions it yields the correct result, and (iii) give the correct formulation.

Solution by the proposers.

Let $s_i = m_i - 1$, and put

$$Y(t) = \sum_{n=0}^{\infty} y_n \frac{t^n}{n!} = \sum_{i=1}^{m} e^{r_i t} \sum_{j=0}^{s_i} b_{ij} t^j .$$

Now define $n_{(s)} = n(n-1)\cdots(n-s+1)$, $n_{(0)} = 1$, and for $k = 1, 2, \cdots, m$ let

$$Y_k(t) = e^{r_k t} \sum_{j=0}^{s_k} b_{kj} t^j$$

so that

$$Y_k(t) = \sum_{v=0}^{\infty} \sum_{j=0}^{s_k} b_{kj} \frac{r_k^v t^{v+j}}{v!}$$

$$= \sum_{v=0}^{\infty} \sum_{j=0}^{s_k} b_{kj} r_k^v (v+j)_{(j)} \frac{t^{v+j}}{(v+j)!} .$$

For $p = 0, 1, \cdots, s_k$ put

$$Y_{k,p}(t) = \sum_{v=0}^{\infty} b_{kp} r_k^v (v+p)_{(p)} \frac{t^{v+p}}{(v+p)!}$$

Differentiating this n times and setting $t = 0$,

$$Y_{k,p}^{(n)}(0) = \sum_{v=0}^{\infty} b_{kp} r_k^v (v+p)_{(p)} \left.\frac{t^{v+p-n}}{(v+p-n)!}\right|_{t=0} = b_{kp} r_k^{n-p} n_{(p)} = r_k^n (b_{kp} n_{(p)} r_k^{-p}) .$$

Thus applying the inverse transform (3.3), we find

$$y_n = Y^{(n)}(0) = \sum_{k=1}^{m} \sum_{p=0}^{s_k} Y_{k,p}^{(n)}(0) = \sum_{i=1}^{m} r_i^n \sum_{j=0}^{m_i-1} b_{ij} n_{(j)} r_i^{-j}$$

which is the correct form.

(ii) If $m_i = 1$ $(i = 1, 2, \cdots, m)$, then since $n_{(0)} = n^0$, Jeske's form gives the correct result. Also, since $n_{(1)} = n^1$, his result will be correct if all roots of multiplicity two are one, and there are no roots of greater multiplicity. For higher multiplicities his form almost never gives the correct result.

(i) We need only take a recurrence whose auxiliary equation does not satisfy the conditions of (ii) to form a counterexample to (*).

Also solved by P. F. Byrd and D. Zeitlin.

Editorial Comment: The b_{ij} in the first displayed equation above are arbitrary constants. The b_{ij} in the second displayed equation are also arbitrary constants. In this sense Jeske is correct. However, most readers would probably incorrectly infer that after you have determined the specific constants for a given problem one can then use these in the second displayed equation which, of course, is not true in all cases. V. E. H.

AN INTERESTING ANGLE

H-67 Proposed by J. W. Gootherts, Sunnyvale, California.

Let $B = (B_0, B_1, \cdots, B_n)$ and $V = (F_m, F_{m+1}, \cdots, F_{m+n})$ be two vectors in Euclidian $n + 1$ space. The B_i's are binomial coefficients of degree n and the F_{m+i}'s are consecutive Fibonacci numbers starting at any integer m.

Find the limit of the angle between these vectors as n approaches infinity.

Solution by F. D. Parker, Sony at Buffalo, N.Y.

We start with the formula

$$\cos^2\theta = \frac{(B \cdot V)^2}{|B|^2 |V|^2},$$

where $B \cdot V$ is the scalar product of B and V, and $|B|$, $|V|$ represent the magnitudes of B and V, respectively.

The following results are easy to verify by mathematical induction:

(1) $$B \cdot V = F_{m+2n}$$

(2) $$|B| = \sqrt{\binom{2n}{n}}$$

(3) $$|V| = \sqrt{F_{m+n}F_{m+n+1} - F_{m-1}F_m}$$

Thus

$$\frac{|B\cdot V|^2}{|B|^2|V|^2} = \frac{(F_{m+2n})^2}{\binom{2n}{n}(F_{m+n}F_{m+n+1} - F_{m-1}F_m)} .$$

But

$$\lim_{n\to\infty} \frac{(F_{m+2n})^2}{F_{m+n}F_{m+n+1} - F_{m-1}F_m} = 0 \quad , \qquad \text{and}$$

$$\lim_{n\to\infty}\binom{2n}{n} = \infty, \text{ and hence } \lim_{n\to\infty} \cos\theta = 0, \text{ and}$$

$$\lim_{n\to\infty} \theta = \pi/2 .$$

Also solved by the proposer.

MANY ROADS TO MORGANTOWN

H-68 Proposed by H. W. Gould, W. Virginia University, Morgantown, W. Va.

Prove that

$$\sum_{k=1}^{n} \frac{1}{F_k} \geq \frac{n^2}{F_{n+2} - 1} , \qquad n \geq 1$$

with equality only for $n = 1, 2$.

Solution by the proposer.

The well-known identity

$$\sum_{i=1}^{n} A_i \sum_{j=1}^{n} B_j = n\sum_{i=1}^{n} A_iB_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (A_i - A_j)(B_j - B_i)$$

yields the special case

$$\sum_{i=1}^{n} A_i \sum_{j=1}^{n} \frac{1}{A_j} = n^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{(A_i - A_j)^2}{A_iA_j} ,$$

whence it is evident that for positive A's we have the inequality

$$\sum_{i=1}^{n} A_i \sum_{j=1}^{n} \frac{1}{A_j} \geq n^2$$

with equality only when $A_i = A_j$ for all $1 \le i \le n$, $1 \le j \le n$. The application to the Fibonacci numbers F_n (with $F_{n+1} = F_n + F_{n-1}$ and $F_1 = 1$, $F_2 = 1$) is evident from the formula

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 ,$$

so that we find

$$\sum_{i=1}^{n} \frac{1}{F_i} \ge \frac{n^2}{F_{n+2} - 1} ,$$

with equality only for $n = 1, 2$.

Zeitlin and Desmond used the Arithmetic-Harmonic mean inequality. Brown used the Schwarz inequality.

Further results are:

$$\sum_{k=1}^{n} \frac{1}{H_k} \ge \frac{n^2}{H_{n+2} - H_2} , \quad n \ge 1 \qquad \text{(Zeitlin)}$$

$$\sum_{k=1}^{n} \frac{1}{F_k^2} \ge \frac{n^2}{F_n F_{n+1}} , \quad n \ge 1 \qquad \text{(Hoggatt)}$$

Also solved by D. Zeitlin, John L. Brown, Jr., M.N.S. Swamy, D. Lind, C.B.A. Peck, and John Wessner.

SOME BELATED SOLVERS' CREDITS

H-37 Dermott A. Breault

H-48 John L. Brown, Jr., and Charles R. Wall

H-52 C.B.A. Peck, F. D. Parker, and D. Lind

H-57 John L. Brown, Jr., Charles R. Wall, Marjorie Bicknell, F.D. Parker, and M.N.S. Swamy

H-58 David Klarner

H-74 John L. Brown, Jr.

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Continued from page 44.

REFERENCE

1. S. L. Basin and V. E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence — Part II," Fibonacci Quarterly, Vol. 1 (1963), No. 2, 61-68.

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