# SUMMATION OF $\sum_{k=1}^{n} k^m F_{k+r}$ FINITE DIFFERENCE APPROACH

BROTHER ALFRED BROUSSEAU, St. Mary's College, California

Let it be proposed to discover an expression for the summation

$$\sum_{k=1}^{n} k^m F_k$$

or more generally

$$\sum_{k=1}^{n} k^m F_{k+r}$$

where m and r are positive integers. One possible approach is a modified version of finite differences. Given an expression f(n) where n is a positive integer, the usual finite difference relation is

$$\Delta f(n) = f(n + 1) - f(n)$$

The adapted finite difference pertains to a quantity of the form

$$f\left[n,\ F_{(n)}\right]$$

where f is a function of n and Fibonacci numbers involving n in their subscripts. We shall define

$$\Delta f\left[n,\ F_{(n)}\right] = f\left[(n + 1),\ F_{(n+1)}\right] - f\left[n, F_{(n)}\right]$$

For example,

$$\Delta(n^2F_n) = (n + 1)^2F_{n+1} - n^2F_n$$
$$= n^2F_{n-1} + (2n + 1)F_{n+1}$$

Likewise we define $\Delta^{-1}$ to be the inverse of $\Delta$ so that

$$\Delta^{-1}[n^2 F_{n-1} + (2n + 1)F_{n+1}] = n^2 F_n + C$$

where there is an arbitrary summation constant C which may involve Fibonacci numbers but these as well as other constituent elements must be free of n.

For our purposes it turns out to be more convenient to seek the value of

$$\sum_{k=1}^{n-1} k^m F_{k+r}$$

Let this summation be denoted by $\phi[n, F_{(n)}]$. Then

$$\Delta\phi[n, F_{(n)}] = \sum_{k=1}^{n} k^m F_{k+r} - \sum_{k=1}^{n-1} k^m F_{k+r} = n^m F_{n+r}$$

Thus

$$\phi[n, F_{(n)}] = \sum_{k=1}^{n-1} k^m F_{k+r} = \Delta^{-1}(n^m F_{n+r})$$

We need then simply to evaluate this inverse finite difference in order to obtain an expression for the summation.

We develop certain relations for this purpose.

(1) $\Delta(n F_{n+r+1}) = (n + 1)F_{n+r+2} - n F_{n+r+1} = n F_{n+r} + F_{n+r+2}$

(2) $\Delta(n^2 F_{n+r+1}) = n^2 F_{n+r} + (2n + 1)F_{n+r+2} = n^2 F_{n+r} + \Delta(n^2) F_{n+r+2}$

and in general

(3) $\Delta(n^m F_{n+r+1}) = n^m F_{n+r} + \Delta(n^m) F_{n+r+2}$

Using formula (1)

$$(4) \qquad \Delta^{-1}(nF_{n+r}) = nF_{n+r+1} - \Delta^{-1}(F_{n+r+2}) = nF_{n+r+1} - F_{n+r+3} + C\,.$$

Then from this result and (2)

$$\Delta^{-1}(n^2F_{n+r}) = n^2F_{n+r+1} - (2n+1)F_{n+r+3} + 2F_{n+r+5} + C$$
$$= n^2F_{n+r+1} - \Delta(n^2)F_{n+r+3} + \Delta^2(n^2)F_{n+r+5} + C$$

The general formula that suggests itself is

$$(6) \qquad \Delta^{-1}(n^mF_{n+r}) = n^mF_{n+r+1} - \Delta(n^m)F_{n+r+3} + \Delta^2(n^m)F_{n+r+5} + \cdots$$
$$= \sum_{t=0}^{m} (-1)^t\Delta^t(n^m)F_{n+r+2t+1} + C$$

That this result is correct may be shown by calculating

$$\Delta[\Delta^{-1}(n^mF_{n+r})]$$

from the summation in (6). The result is $n^mF_{n+r}$ as can be readily seen from the fact that apart from the first term in the expansion all succeeding terms cancel in pairs. The results for the first two terms will show the pattern.

$$\Delta(n^mF_{n+r+1}) = n^mF_{n+r} + \Delta(n^m)F_{n+r+2} \qquad \text{by (3)}$$
$$\Delta[-\Delta(n^m)F_{n+r+3}] = -\Delta(n+1)^mF_{n+r+4} + \Delta(n^m)F_{n+r+3}$$
$$= -\Delta(n^m)F_{n+r+4} - \Delta^2(n^m)F_{n+r+4} + \Delta(n^m)F_{n+r+3}$$
$$= -\Delta(n^m)F_{n+r+2} - \Delta^2(n^m)F_{n+r+4} \quad .$$

Hence (6) provides the required formula apart from making explicit the coefficients in terms of $n$ and calculating the undetermined constant. The former are given subsequently in tables; the latter may be obtained as shown below for the particular case in which $m = 5$.

We set $n = 2$ in (6) so that

$$F_{r+1} = 32F_{r+3} - 211F_{r+5} + 570F_{r+7} - 750F_{r+9} + 480F_{r+11} - 120F_{r+13} + C$$

Using the formulas

$$F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}$$

and

$$F_n = (-1)^{k-1}(F_kF_{n+k+1} - F_{k+1}F_{n+k})$$

$C$ is found to be $16679\,F_{r+9} + 10324\,F_{r+8}$ .

Table 1
COEFFICIENTS OF $\Delta(n^m)$

| m | 1 | n | $n^2$ | $n^3$ | $n^4$ | $n^5$ | $n^6$ | $n^7$ | $n^8$ | $n^9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | |
| 2 | 1 | 2 | | | | | | | | |
| 3 | 1 | 3 | 3 | | | | | | | |
| 4 | 1 | 4 | 6 | 4 | | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | | | | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | | | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | | | |
| 8 | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | | |
| 9 | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | |
| 10 | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 |

Table 2
COEFFICIENTS OF $\Delta^2(n^m)$

| m | 1 | n | $n^2$ | $n^3$ | $n^4$ | $n^5$ | $n^6$ | $n^7$ | $n^8$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | | | | | | | | |
| 3 | 6 | 6 | | | | | | | |
| 4 | 14 | 24 | 12 | | | | | | |
| 5 | 30 | 70 | 60 | 20 | | | | | |
| 6 | 62 | 180 | 210 | 120 | 30 | | | | |
| 7 | 126 | 434 | 630 | 490 | 210 | 42 | | | |
| 8 | 254 | 1008 | 1736 | 1680 | 980 | 336 | 56 | | |
| 9 | 510 | 2286 | 4536 | 5208 | 3780 | 1764 | 504 | 72 | |
| 10 | 1022 | 5100 | 11430 | 15120 | 13020 | 7560 | 2940 | 720 | 90 |

Table 3

COEFFICIENTS OF $\Delta^3(n^m)$

| m | 1 | n | $n^2$ | $n^3$ | $n^4$ | $n^5$ | $n^6$ | $n^7$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 6 | | | | | | | |
| 4 | 36 | 24 | | | | | | |
| 5 | 150 | 180 | 60 | | | | | |
| 6 | 540 | 900 | 540 | 120 | | | | |
| 7 | 1806 | 3780 | 3150 | 1260 | 210 | | | |
| 8 | 5796 | 14448 | 15120 | 8400 | 2520 | 336 | | |
| 9 | 18150 | 52164 | 65016 | 45360 | 18900 | 4536 | 504 | |
| 10 | 55980 | 181500 | 260820 | 216720 | 113400 | 37800 | 7560 | 720 |

Table 4

COEFFICIENTS OF $\Delta^4(n^m)$

| m | 1 | n | $n^2$ | $n^3$ | $n^4$ | $n^5$ | $n^6$ |
|---|---|---|---|---|---|---|---|
| 4 | 24 | | | | | | |
| 5 | 240 | 120 | | | | | |
| 6 | 1560 | 1440 | 360 | | | | |
| 7 | 8400 | 10920 | 5040 | 840 | | | |
| 8 | 40824 | 67200 | 43680 | 13440 | 1680 | | |
| 9 | 186480 | 367416 | 302400 | 131040 | 30240 | 3024 | |
| 10 | 818520 | 1864800 | 1837080 | 1008000 | 327600 | 60480 | 5040 |

Table 5

COEFFICIENTS OF $\Delta^5(n^m)$

| m | 1 | n | $n^2$ | $n^3$ | $n^4$ | $n^5$ |
|---|---|---|---|---|---|---|
| 5 | 120 | | | | | |
| 6 | 1800 | 720 | | | | |
| 7 | 16800 | 12600 | 2520 | | | |
| 8 | 126000 | 134400 | 50400 | 6720 | | |
| 9 | 834120 | 1134000 | 604800 | 151200 | 15120 | |
| 10 | 5103000 | 8341200 | 5670000 | 2016000 | 378000 | 30240 |

Table 6
COEFFICIENTS OF $\Delta^6(n^m)$

| m | 1 | n | $n^2$ | $n^3$ | $n^4$ |
|---|---|---|---|---|---|
| 6 | 720 | | | | |
| 7 | 15120 | 5040 | | | |
| 8 | 191520 | 120960 | 20160 | | |
| 9 | 1905120 | 1723680 | 544320 | 60480 | |
| 10 | 16435440 | 19051200 | 8618400 | 1814400 | 151200 |

Table 7
COEFFICIENTS OF $\Delta^7(n^m)$

| m | 1 | n | $n^2$ | $n^3$ |
|---|---|---|---|---|
| 7 | 5040 | | | |
| 8 | 141120 | 40320 | | |
| 9 | 2328480 | 1270080 | 181440 | |
| 10 | 29635200 | 23284800 | 6350400 | 604800 |

Table 8
COEFFICIENTS OF $\Delta^8(n^m)$

| m | 1 | n | $n^2$ |
|---|---|---|---|
| 8 | 40320 | | |
| 9 | 1451520 | 362880 | |
| 10 | 30240000 | 14515200 | 1814400 |

Table 9
COEFFICIENTS OF $\Delta^9(n^m)$

| m | 1 | n |
|---|---|---|
| 9 | 362880 | |
| 10 | 16329600 | 3628800 |

Table 10
COEFFICIENTS OF $\Delta^{10}(n^m)$

| m | 1 |
|---|---|
| 10 | 3628800 |

Table 11
SUMMATION CONSTANTS

| m | Summation Constants |
|---|---|
| 1 | $F_{r+3}$ |
| 2 | $-F_{r+6}$ |
| 3 | $7F_{r+5} + 5F_{r+4}$ |
| 4 | $-37F_{r+6} - 24F_{r+5}$ |
| 5 | $242F_{r+7} + 147F_{r+6}$ |
| 6 | $-1861F_{r+8} - 1139F_{r+7}$ |
| 7 | $16679F_{r+9} + 10324F_{r+8}$ |
| 8 | $-171362F_{r+10} - 106089F_{r+9}$ |
| 9 | $1981723F_{r+11} + 1224729F_{r+10}$ |
| 10 | $-25453505F_{r+12} - 15726832F_{r+11}$ |

To be able to write out a complete formula one uses formula (6) and the various tables. The case $m = 7$ is given below.

$$\begin{aligned}\sum_{k=1}^{n} k^7 F_{k+r} &= n^7 F_{n+r+1} - (7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n + 1)F_{n+r+3} \\ &+ (42n^5 + 210n^4 + 490n^3 + 630n^2 + 434n + 126)F_{n+r+5} \\ &- (210n^4 + 1260n^3 + 3150n^2 + 3780n + 1806)F_{n+r+7} \\ &+ (840n^3 + 5040n^2 + 10920n + 8400)F_{n+r+9} \\ &- (2520n^2 + 12600n + 16800)F_{n+r+11} + (5040n + 15120)F_{n+r+13} \\ &- 5040F_{n+r+15}\end{aligned}$$

## CALCULATION BY FINITE DIFFERENCES

Except for the smaller values of $m$, the explicit formulas given above in terms of $n$ are apt to involve undue calculations. These can be obviated by going directly to finite differences and using formula (6).

For example, to calculate

$$\sum_{k=1}^{49} k^5 F_{k+7}$$

we would first write down the values of $k^5$ for $k = 50, 51$, etc.

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| k | $k^5$ |
|---|---|
| 50 | 312500000 |
| 51 | 345025251 |
| 52 | 380204032 |
| 53 | 418195493 |
| 54 | 459165024 |
| 55 | 503284375 |

Then

$$\Delta\,[k^5]_{k=50} = 34502521 - 312500000 = 32525251$$

$$\Delta^2[k^5]_{k=50} = 380204032 - 2 \cdot 34502521 + 312500000 = 2653530$$

$$\Delta^3[k^5]_{k=50} = 418195493 - 3 \cdot 380204032 + 3 \cdot 345025251 - 312500000$$
$$= 159150$$

$$\Delta^4[k^5]_{k=50} = 459165024 - 4 \cdot 418195493 + 6 \cdot 380204032 - 4 \cdot 345025251$$
$$+ 312500000 = 6240$$

$$\Delta^5[k^5]_{k=50} = 120$$

The value of the summation is:

$$312500000\,F_{58} - 32525251\,F_{60} + 2653530\,F_{62} - 159150\,F_{64} + 6240\,F_{66} - 120\,F_{68}$$
$$+ 242\,F_{14} + 147F_{13}$$

which can either be calculated directly or sum of the terms can be unified and the number of multiplications of large numbers can be decreased.

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