ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A. P. HILLMAN, University of New Mexico, Albuquerque, New Mex.

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-106 Proposed by H. H. Ferns, Victoria, B.C., Canada.

Prove the following identities:

$$2F_{i+j} = F_iL_j + F_jL_i ,$$
$$2L_{i+j} = L_iL_j + 5F_iF_j .$$

B-107 Proposed by Robert S. Seamons, Yakima Valley College, Yakima, Wash.

Let M_n and G_n be respectively the n^{th} terms of the sequences (of Lucas and Fibonacci) for which $M_n = M_{n-1}^2 - 2$, $M_1 = 3$, and $G_n = G_{n-1} + G_{n-2}$, $G_1 = 1$, $G_2 = 2$. Prove that

$$M_n = 1 + [\sqrt{5}\,G_m] ,$$

where $m = 2^n - 1$ and $[x]$ is the greatest integer function.

B-108 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let $u_1 = p$, $y_2 = q$, and $u_{n+2} = u_{n+1} + u_n$. Also let $S_n = u_1 + u_2 + \cdots + u_n$. It is true that $S_6 = 4u_4$ and $S_{10} = 11u_7$. Generalize these formulas.

B-109 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let r and s be the roots of the quadratic equation $x^2 - px - q = 0$, $(r \neq s)$. Let $U_n = (r^n - s^n)/(r - s)$ and $V_n = r^n + s^n$. Show that

$$V_n = U_{n+1} + qU_{n-1} .$$

B-110 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2n+1}}$$

B-111 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{F_{4n+2}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{1}{L_{4n+2}}$$

SOLUTIONS

LUCAS NUMBERS MODULO 5

B-88 Proposed by John Wessner, Melbourne, Florida.

Let L_0, L_2, L_4, $L_6, \cdots$ be the Lucas numbers 2, 3, 7, 18, $\cdots$. Show that

$$L_{2k} \equiv 2(-1)^k \pmod 5 .$$

Solution by J. A. H. Hunter, Toronto, Canada

All (mod 5) we have: $L_1 \equiv 1$, $L_2 \equiv -2$, $L_3 \equiv -1$, $L_4 \equiv 2$, $L_5 \equiv 1$, $L_6 \equiv -2$, etc., so it follows that $L_{4t+2} \equiv -2$ and $L_{4t} \equiv +2$. Hence $L_{2k} \equiv 2(-1)^k \pmod 5$.

Also solved by James E. Desmond, H. H. Ferns, Joseph D. E. Konhauser, Douglas Lind, F. D. Parker, C.B.A. Peck, Jeremy C. Pond, David Zeitlin, and the proposer.

A CLOSE APPROXIMATION

B-89 Proposed by Robert S. Seamons, Yakima Valley College, Yakima, Wash.

Let F_n and L_n be the n^{th} Fibonacci and n^{th} Lucas numbers, respectively. Let $[x]$ be the greatest integer function. Show that $L_{2m} = 1 + [\sqrt{5}F_{2m}]$ for all positive integers m.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

From the Binet forms for F_n and L_n, the statement is equivalent to $\alpha^{2m} + \beta^{2m} = [1 + \alpha^{2m} - \beta^{2m}]$, where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$. But $1/2 > \beta^{2m} > 0$ for $m > 0$, so we have

$$\alpha^{2m} + \beta^{2m} \le \alpha^{2m} - \beta^{2m} + 1 < \alpha^{2m} + \beta^{2m} + 1$$

which implies $\alpha^{2m} + \beta^{2m} = [1 + \alpha^{2m} - \beta^{2m}]$, as desired.

Also solved by James E. Desmond, H. H. Ferns, C.B.A. Peck, Jeremy C. Pond, David Zeitlin, and the proposer.

B-90 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let $b_1, b_2, \cdots$ be the sequence 3, 7, 47, $\cdots$ with recurrence relation $b_{n+1} = b_n^2 - 2$. Show that the roots of

$$x^2 - 2b_n x + 4 = 0$$

are expressible in the form $c + d\sqrt{5}$, where c and d are integers.

Solution by David Zeitlin, Minneapolis, Minnesota.

The roots are $b_n \pm \sqrt{b_n^2 - 4} = b_n \pm \sqrt{b_{n+1} - 2}$. The recursion relation may be written as $U_{n+1} = (b_n + 2)U_n$, where $U_n = b_n - 2$, $U_1 = 1$. Thus,

$$\frac{U_{n+1}}{U_1} = \prod_{k=1}^{n} \frac{U_{k+1}}{U_k} = \prod_{k=1}^{n} (b_k + 2) = 5 \prod_{k=2}^{n} (b_k + 2)$$

$$= 5 \prod_{j=1}^{n-1} (b_{j+1} + 2) = 5 \prod_{j=1}^{n-1} b_j^2 ,$$

or

$$b_{n+1} - 2 = 5 \prod_{j=1}^{n-1} b_j^2 .$$

Thus,

$$c = b_n, \quad \text{and} \quad d = \pm \prod_{j=1}^{n-1} b_j , \quad n = 2, 3, \cdots .$$

Also solved by James E. Desmond, H. H. Ferns, Douglas Lind, C.B.A. Peck, Jeremy C. Pond, John Wessner, and the proposer.

CONVERGENCE OF SERIES

B-91 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

If F_n is the n^{th} Fibonacci number, show that

$$\sum_{j=1}^{\infty} (1/F_j)$$

converges while

$$\sum_{j=3}^{\infty} (1/\ln F_j)$$

diverges.

Solution by Jeremy C. Pond, Sussex, England.

$$\left(\frac{1}{F_n}\right) \bigg/ \left(\frac{1}{F_{n=1}}\right) = \frac{F_{n+1}}{F_n} \rightarrow \frac{1+\sqrt{5}}{2} > 1 \quad \text{as} \quad n \rightarrow \infty$$

and so

$$\sum_{j=1}^{\infty} (1/F_j)$$

converges by d'Alembert's test. Also,

$$(1/\ln F_n)/(1/n) \rightarrow 1/\ln\left(\frac{1+\sqrt{5}}{2}\right) > 0$$

and so $1/\ln F_j$ and $1/n$ diverge together.

Also solved by C. B. A. Peck and the proposer.

GREATEST COMMON DIVISOR

B-92 Proposed by J. L. Brown, Jr., The Pennsylvania State University.

Let (x,y) denote the g.c.d of positive integers x and y. Show that $(F_m, F_n) = (F_m, F_{m+n}) = (F_n, F_{m+n})$ for all positive integers m and n.

I. Solution by Joseph D. E. Konhauser, Univ. of Minnesota, Minneapolis, Minn.

We use the well-known identity

$$F_{m+n} = F_{n-1}F_m + F_nF_{m+1}$$

and the fact that two consecutive Fibonacci numbers are relatively prime.

Let $d = (F_m, F_n)$ then, from (1) $d|F_{m+n}$. Let $e = (F_{m+n}, F_m)$ then, from (1), $e|F_n$, since $(F_m, F_{m+1}) = 1$. On the one hand, $e|d$ (since $e|F_m$ and $e|F_n$). On the other hand, $d|e$ (since $d|F_m$ and $d|F_{m+n}$). Therefore, $d = e$; that is, $(F_m, F_n) = (F_m, F_{m+n})$. In like manner, it follows that $(F_m, F_n) = (F_n, F_{m+n})$.

II. Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

It is well known [N. N. Vorobyov, The Fibonacci Numbers, page 23, Theorem 4] that $(F_m, F_n) = F_{(m,n)}$. The desired result then follows immediately from the easily established fact that $(m,n) = (m, m+n) = (n, m+n)$.

Also solved by Thomas P. Dence, James E. Desmond, John E. Homer, Jr., C.B.A. Peck, Jeremy C. Pond, David Zeitlin, and the proposer.

L_n MODULO n

B-93 Proposed by Martin Pettet, Toronto, Ontario, Canada

Show that if n is a positive prime, $L_n \equiv 1 \pmod{n}$. Is the converse true?

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia.

From the Binet form we have

$$L_n = 2^{-n}\{(1+\sqrt{5})^n + (1-\sqrt{5})^n\} = 2^{-n}\left(\sum_{j=0}^{n} \binom{n}{j} 5^{j/2}\{1+(-1)^j\}\right)$$
$$= 2^{-n+1}\sum_{j=0}^{[n/2]} \binom{n}{2j} 5^j ,$$

where $[x]$ denotes the greatest integer contained in x. Now if n is prime,

$$\binom{n}{2j} \equiv 0 \pmod{n} \quad (j = 1, 2, \cdots, n/2),$$

so that

$$L_n \equiv \binom{n}{0} \Big/ 2^{n-1} \equiv 1/2^{n-1} \pmod{n}.$$

By Fermat's Lesser Theorem, $2^{n-1} \equiv 1 \pmod{n}$, so that $L_n \equiv 1 \pmod{n}$ if n is prime.

I have not been able to prove or disprove the converse of this statement. A calculation by computer indicates that the converse is true for $n < 700$.

Also solved by the proposer who stated that the converse is false and gave 705, 2465, and 2737 as the first few composite values of n.

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NOTICE

George Ledin, Jr. has been appointed by The Fibonacci Association to collect and classify all existing Fibonacci Identities, Lucas Identities, and Hybrid Identities. We request that readers send copies of their private lists (with possible reference sources) to

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for inclusion in the planned booklet.

Verner E. Hoggatt, Jr.,
Director

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