RELATIONS INVOLVING LATTICE PATHS AND CERTAIN SEQUENCES OF INTEGERS

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Relations involving certain special planar lattice paths and certain sequences of integers have been studied previously [1], [2]. We will state certain basic definitions which pertain to these studies, develop additional results involving other planar lattice paths, and finally, indicate generalizations of these results for lattice paths in k dimensional space. For convenience of reference some of the definitions are collected together and presented in Part 1. The remaining material will be found in Part 2.

Part 1

In Euclidean k-dimensional space the set X of points such that p belongs to X if and only if each coordinate of p is an integer is called the unit lattice of that space.

The statement that P is a lattice path in a certain space means that P is a sequence such that

1) each term of P is a member of the unit lattice of that space, and
2) if X is a term of P and Y is the next term of P and x_i and y_i are the i^{th} coordinates of X and Y respectively, then $|x_i - y_i| =$ 1 or 0 and for some j, $|x_j - y_j| = 1$.

If each of X and Y is a point of the unit lattice in Euclidean k-dimensional space, then the statement that the lattice path P is a path from X to Y means that P is finite, X is the first term of P, and Y is the last term of P. If P is a lattice path, X is a term of P, and Y is the next term of P, then by the step [X, Y] of P is meant the line interval whose end points are X and Y.

A lattice path P in Euclidean 2 or 3-space is said to be symmetric with respect to the line k if and only if it is true that if X is a point of some step of P, then either X is a point of k or there exists a point Y of some step of P such that k is the perpendicular bisector of the line interval [X, Y].

Suppose that $S = [(x_1,y_1), (x_2,y_2)]$ is a step of some lattice path P in Euclidean 2-space. S is said to be x-increasing if $x_2 - x_1 = 1$ and x-decreasing if $x_2 - x_1 = -1$. The terms y-increasing and y-decreasing are similarly defined. A step is said to be xy-increasing if it is both x-increasing and y-increasing. To say that S is x-increasing only means that S is x-increasing but neither y-increasing nor y-decreasing. P is said to be x-monotonically increasing if and only if it is true that if Σ is a step of P, then Σ is not x-decreasing. The term y-monotonically increasing is similarly defined. A step Σ is said to be vertical if it is neither x-increasing nor x-decreasing. A step Σ is said to be horizontal if it is neither y-increasing nor y-decreasing. The statement that the path P is duotonically increasing means that P is both x-monotonically increasing and y-monotonically increasing.

Part 2

In Euclidean 2-space a path from (0,0) to (n,n) is said to have property G if and only if:

1) it is duotonically increasing,
2) it is symmetric with respect to the line $x + y = n$, and
3) no step of it which contains a point below the line $x + y = n$ is vertical.

A path having property G will be called a G-path.

Theorem 1 (Greenwood)

Let $g(0) = 1$ and $g(1) = 1$. For each positive integer $n \geq 2$, let $g(n)$ denote the number of G-paths from (0,0) to $(n - 1, n - 1)$. The sequence $\{g(0), g(1), \cdots, g(n), \cdots\}$ is the Fibonacci sequence.

Proof. By definition $g(0) = g(1) = 1$. Suppose $n = 2$. The only G-paths from (0,0) to (1,1) are $\{(0,0), (1,0), (1,1)\}$ and $\{(0,0), (1,1)\}$, thus $g(2) = 2$. For $n = 3$, the G-paths from (0,0) to (2,2) are $\{(0,0), (1,0), (2,0), (2,1), (2,2)\}$, $\{(0,0), (1,0), (2,1), (2,2)\}$ and $\{(0,0), (1,1), (2,2)\}$, so that $g(3) = 3$.

Suppose $n \geq 4$. Each G-path from (0,0) to $(n - 1, n - 1)$ has as its initial step either $[(0,0), (1,0)]$ or $[(0,0), (1,1)]$. If a G-path has as its initial step $[(0,0), (1,0)]$, then, because of symmetry, its terminal step is $[(n - 1, n - 2), (n - 1, n - 1)]$; and thus it contains as a subsequence a G-path from (1,0) to $(n - 1, n - 2)$. But the number of G-paths from (1,0) to $(n - 1, n - 2)$ is the number of G-paths from (0,0) to $(n - 2, n - 2)$, i.e., $g(n - 1)$.

Likewise, if a G-path has as its initial step $[(0,0), (1,1)]$, then its terminal step is $[(n - 2, n - 2), (n - 1, n - 1)]$, and it contains as a subsequence a G-path from (1,1) to (n - 2, n - 2). The number of such G-paths is the number of G-paths from (0,0) to (n - 3, n - 3), which is $g(n - 2)$. Thus $g(n) = g(n - 1) + g(n - 2)$.

The statement that a path in Euclidean 2-space has property H means that it has property G and is such that one of its terms belongs to the line $x + y = n$. A path having property H will be called an H-path.

Obviously, if n is a positive integer, then the set of all H-paths from (0,0) to (n,n) is a proper subset of the set of all G-paths from (0,0) to (n,n); yet, using an argument similar to the above, we may establish the following.

Theorem 2.

Let $h(0) = 1$ and, for each positive integer n, let $h(n)$ denote the number of H-paths from (0,0) to (n,n). The sequence $\{h(0), h(1), \cdots, h(n), \cdots\}$ is the Fibonacci sequence.

An obvious but interesting corollary is that the number of H-paths from (0,0) to (n,n) is the number of G-paths from (0,0) to (n - 1, n - 1).

Greenwood has discussed G-paths [1]. A method of enumeration different from that used by Greenwood leads to the following [2].

Theorem 3.

Let

$$z(1,i) = 1,$$

$$z(2,i) = \left[\frac{i-1}{2}\right], \text{ where } [\,] \text{ denotes the greatest integer function,}$$

$$z(3,i) = z(3,i-1) + z(2,i-1) ,$$

$$z(4,i) = z(4,i-2) + z(3,i-2) ,$$

$$\cdots$$

$$z(2n,i) = z(2n,i-2) + z(2n-1,i-2) ,$$

$$z(2n+1,i) = z(2n+1,i-1) + z(2n,i-1) ,$$

$$\cdots$$

with the restriction that $z(k,i) = 0$ if $k > i$. For each positive integer i, let

$$f(i) = \sum_{k=1}^{i} z(k,i) \quad .$$

The sequence $\{f(i) | i = 1, 2, \cdots\}$ is the Fibonacci sequence.

The proof is direct and is omitted. A geometric interpretation of the numbers $z(k,i)$ and $f(i)$ is given in [2].

It is interesting to note the sequence obtained by considering paths in 3-space that are analogous to H-paths in 2-space. In Euclidean 3-space, a path from $(0,0,0)$ to (n,n,n) is said to have property F if and only if it is such that:

1) it is symmetric with respect to the line $z = (n/2)$ in the plane $x + y = n$,
2) if the step $[P_1, P_2]$ of it is z-increasing only, then P_1 belongs to the plane $x + y = n$,
3) if S is a step of it which is not z-increasing only, then either S is x-increasing only, y-increasing only, or xyz-increasing, and
4) some term of it belongs to the plane $x + y = n$.

We will call a path an F-path if it has a property F.

We define $f(0) = 1$; and, for each positive integer n, let $f(n)$ denote the number of F-paths from $(0,0,0)$ to (n,n,n). We note that $f(1) = 2$ and $f(2) = 5$. If $n > 2$, then each F-path has as its second term either $(1,0,0)$, $(0,1,0)$, or $(1,1,1)$. If an F-path from $(0,0,0)$ to (n,n,n) has as its second term $(1,0,0)$ or $(0,1,0)$, then it has as its next to last term $(n, n-1, n)$ or $(n-1, n, n)$ respectively. The number of F-paths from $(0,0,0)$ to (n,n,n) which have as their second term either $(0,1,0)$ or $(1,0,0)$ is the number of F-paths from $(0,0,0)$ to $(n-1, n-1, n-1)$. Hence, the number of F-paths from $(0,0,0)$ to (n,n,n) whose second term is either $(1,0,0)$ or $(0,1,0)$ is $2f(n-1)$. Similarly, the number of F-paths from $(0,0,0)$ to (n,n,n) whose second term is $(1,1,1)$ is $f(n-2)$. Hence, if $n > 2$, then $f(n) = 2f(n-1) + f(n-2)$.

It is noted that the expression $f(n) = 2f(n-1) + f(n-2)$ is the special case of the Fibonacci polynomial $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$ for $f_0(x) = 0$, $f_1(x) = 1$, and $x = 2$.

Using the methods of finite difference equations we may obtain an expression for calculating $f(n)$ directly. Consider again the recursion relation $f(n) = 2f(n-1) + f(n-2)$ in the form of the second order homogeneous difference equation

$$f(n+2) - 2f(n+1) - f(n) = 0 .$$

The corresponding characteristic equation

$$r^2 - 2r - 1 = 0$$

has roots

$$r_1 = 1 + \sqrt{2} \quad \text{and} \quad r_2 = 1 - \sqrt{2} .$$

The general solution of the above difference equation is

$$f(n) = C_1(1 + \sqrt{2})^n + C_2(1 - \sqrt{2})^n .$$

Using the initial conditions of $f(0) = 1$ and $f(1) = 2$, the constants C_1 and C_2 are found to be

$$(\sqrt{2} + 1)/2\sqrt{2} \quad \text{and} \quad (\sqrt{2} - 1)/2\sqrt{2}$$

respectively, so that we have finally

$$f(n) = \frac{(1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1}}{2\sqrt{2}}$$

An analysis similar to that used to obtain the recursion relation for F-paths in 3-space suffices to show that in k-dimensional space the number of paths from $(0,0,0,\cdots,0)$ to $(n,n,n,\cdots,n)$ that are analogous to F paths in 3-space satisfies the recursion relation $f(n) = (k-1)f(n-1) + f(n-k+1)$.

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The references shown below are for "Iterated Fibonacci and Lucas Subscripts," which appears on page 89.

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