NOTE ON A COMBINATORIAL IDENTITY IN THE THEORY OF BI-COLORED GRAPHS

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In connection with an enumeration problem arising in the theory of labelled bi-colored graphs, C. Y. Lee [2] has obtained the following identities. Defining $N(a,b;n)$ by the expansion

$$(1) \qquad \sum_{n=0}^{ab} N(a,b;n)x^n = \sum_{k=0}^{a} (-1)^{a+b+k} \binom{a}{k} \left\{1 - (1+x)^k\right\}^b$$

and noting the lemma

$$(2) \qquad \sum_{j=0}^{b} \sum_{i=0}^{kj} f(i,j) = \sum_{i=0}^{kb} \sum_{j=\langle \frac{i}{k} \rangle}^{b} f(i,j) \quad ,$$

where $\langle x \rangle$ = the smallest integer $\geq x$, Lee was able to show that

$$(3) \qquad N(a,b;n) = \sum_{k=\langle \frac{n}{b} \rangle}^{a} \sum_{j=\langle \frac{n}{k} \rangle}^{b} (-1)^{a+b+k+j} \binom{a}{k}\binom{b}{j}\binom{kj}{n} \quad ,$$

from which as a special case he deduced the apparently novel formula

$$(4) \qquad \sum_{k=1}^{n} \sum_{j=\langle \frac{n}{k} \rangle}^{n} (-1)^{k+j} \binom{n}{k}\binom{n}{j}\binom{kj}{n} = n! \quad .$$

It may be of interest therefore, to point out that the formulas may be written in much simpler form inasmuch as the introduction of $\langle x \rangle$ leads to

*Supported by National Science Foundation Research Grants G-14095 and GP-482.

unnecessarily complicated relations. Indeed it will be shown that relation (4) is essentially trivial and may be generalized by methods of finite differences. The simple nature of relation (4) was also missed in reviews of the paper [4], [5].

In order to determine N from expansion (1) it is not necessary to invoke (2) and instead we shall merely make use of the fact that

$$\binom{m}{p} = 0 \quad \text{for} \quad m < p$$

when m and p are integers with $p \geq 0$, $m \geq 0$.

From (1) we have in fact

$$\sum_{n=0}^{ab} N(a,b;n)x^n = \sum_{k=0}^{a} (-1)^{a+b+k} \binom{a}{k} \sum_{j=0}^{b} (-1)^j \binom{b}{j} (1+x)^{kj}$$

$$= \sum_{k=0}^{a}\sum_{j=0}^{b} (-1)^{a+b+k+j} \binom{a}{k}\binom{b}{j} \sum_{n=0}^{m} \binom{kj}{n} x^n$$

$$= \sum_{n=0}^{m} x^n \sum_{k=0}^{a}\sum_{j=0}^{b} (-1)^{a+b+k+j} \binom{a}{k}\binom{b}{j}\binom{kj}{n}$$

provided only that $m \geq ab$.

Consequently we have

$$N(a,b;n) = \sum_{k=0}^{a}\sum_{j=0}^{b} (-1)^{a+b+k+j} \binom{a}{k}\binom{b}{j}\binom{kj}{n}, \tag{5}$$

without any essential need for the restriction on range of summation in (3). Of course, some terms are zero, but it is convenient to allow these to stand in the indicated formula.

Then when we choose $a = b = n$ in order to obtain the identity (4) found by Lee, we see that this would appear more elegantly in the form

$$(6) \qquad \sum_{k=0}^{n} \sum_{j=0}^{n} (-1)^{k+j} \binom{n}{k}\binom{n}{j}\binom{kj}{n} = n!$$

and we shall show in a simple way that this generalizes to give the relation

$$(7) \qquad \sum_{k=0}^{n} \sum_{j=0}^{n} (-1)^{k+j} \binom{n}{k}\binom{n}{j}\binom{c+kj}{n} = n!$$

for any real value of c.

As for the proof, this comes from the familiar fact that when $f(x)$ is a polynomial of degree $\leq n$ in x, say

$$f(x) = \sum_{i=0}^{n} a_i x^i ,$$

then

$$(8) \qquad \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(k) = \begin{cases} 0, & n < r , \\ (-1)^n n! \, a_n, & r = n . \end{cases}$$

Since $\binom{c+dx}{n}$ is a polynomial of degree n in x we have

$$(9) \qquad \sum_{k=0}^{r} (-1)^k \binom{r}{k}\binom{c+dk}{n} = \begin{cases} 0, & n < r , \\ (-d)^n, & r = n , \end{cases}$$

this being true for all real values of c and d. The identity is not new, and appears for example in Schwatt [3, 104] and has been used by the writer [1] in another connection.

Thus

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{c+kj}{n} = (-1)^n k^n \quad ,$$

so that

$$\sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^{k+j}\binom{n}{k}\binom{n}{j}\binom{c+kj}{n} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^n$$

but this is clearly $n!$ by the familiar Euler theorem about n^{th} differences of n^{th} powers of the natural numbers, or we may again apply (8).

If we define

$$N(a,b,c,n) = \sum_{k=0}^{a}\sum_{j=0}^{b}(-1)^{a+b+k+j}\binom{a}{k}\binom{b}{j}\binom{c+kj}{n} \quad , \tag{10}$$

with $N(a,b,0,n) = N(a,b;n)$, then we have the extension of (1) as

$$\sum_{n=0}^{\infty} N(a,b,c,n)x^n = (1+x)^c\sum_{k=0}^{a}(-1)^{a+b+k}\binom{a}{k}\left\{1-(1+x)^k\right\}^b \quad , \tag{11}$$

and it would be of interest to know whether this yields any interesting result about labelled bi-colored graphs.

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