AMATEUR INTERESTS IN THE FIBONACCI SERIES II
CALCULATION OF FIBONACCI NUMBERS AND SUMS FROM THE BINOMIAL

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As mentioned in an earlier paper, my interest in the Fibonacci series stemmed from the observation (in 1959) that the preferred ratios developed in the research of my colleague, H. Ellner, and later included in Department of Defense Handbook H109 [1], were 1, 2, 3, 5, 8. When the supposition was tested, that all preferred ratios would come from the Fibonacci series, the next ratio was calculated and was found to be 13. Then it was noted that the sample sizes, Acceptable Quality Levels (AQL's), and lot size ranges of all sampling standards since the original work in this field by Dodge and Romig [2] were series approximately of the type:

$$u_{n+2} = u_{n+1} + u_n \tag{1}$$

It seemed self-evident that, in some way, the Fibonacci series must be intimately connected with some probability distribution such as the Binomial expansion. Through a little algebraic juggling such a connection was quickly established as follows:

The method of finite differences, described in Chrystal [3] yields interesting results. If successive differences be taken between adjacent Fibonacci numbers:

$$d_{1,n} = u_{n+1} - u_n \tag{2}$$

a series of first-order differences $d_{1,n}$ is generated. In the same way, a series of second-order differences may be generated:

$$d_{2,n} = d_{1,n+1} - d_{1,n} \tag{3}$$

Higher-order differences may be generated in accordance with the general relationship:

(4) $$d_{k,n} = d_{(k-1),(n+1)} - d_{(k-1),n}$$

Taking the Fibonacci series itself, u_n, to constitute the zero order of differences $d_{0,n}$ and if j is some given value of n, and k is the order of differences, we get the following table:

k	j = 1	2	3	4	5	6	7	8	9	10	11	12	13	14. . . etc.	
d_0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	u_n
d_1	0	1	1	2	3	5	8	13	21	34	55	89	144	233	
d_2	1	0	1	1	2	3	5	8	13	21	34	55	89	144	
d_3	-1	1	0	1	1	2	3	5	8	13	21	34	55	89	
d_4	2	-1	1	0	1	1	2	3	5	8	13	21	34	55	
d_5	-3	2	-1	1	0	1	1	2	3	5	8	13	21	34	
d_6	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21	
d_7	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	
d_8	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	
d_9	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	
d_{10}	34	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	
.															
.															
.															
d_k															

It may be seen that as k, the order of differences, and j are increased without limit, the table of d_k and j forms, both horizontally and vertically, four Fibonacci series centering on each zero, such that the two series, one above and one to the right of each zero are positive in all their terms while the series to the left and below each zero have alternate negative terms. In essence, the latter series constitute the negative branch of the Fibonacci series, u_{-n}.

We can calculate u_n, $n = k+j$, from the differences $d_{k,j}$ as follows:

(5) $$u_{k+j} = d_{0,j} + kd_{1,j} + \frac{k(k-1)}{2!} d_{2,j} + \frac{k(k-1)(k-2)}{3!} d_{3,j} + \cdots + \frac{k!}{k!} d_{k,j}$$

where $d_{k,j}$ is d_j of the order k as shown in the table and $u_j = d_{0,j}$. The coefficients of the $d_{k,j}$ terms represent those of the Binomial Expansion, $(a + b)^k$.

Example 1.

Calculate u_{k+j} when k = 7, and j = 3 $(u_{k+j} = u_{10})$

$$u_{7+3} = d_{0,3} + 7d_{1,3} + \frac{7(6)}{2!} d_{2,3} + \frac{7(6)(5)}{3!} d_{3,3} + \cdots + \frac{7!}{7!} d_{7,3}$$

$$= 2 + 7(1) + \frac{7(6)}{2}(1) + \frac{7(6)(5)}{3(2)}(0) + \frac{7(6)(5)(4)}{4(3)(2)}(1)$$

$$+ \frac{7(6)(5)(4)(3)}{5(4)(3)(2)}(-1) + \frac{7(6)(5)(4)(3)(2)}{(6)(5)(4)(3)(2)}(2) + \frac{7!}{7!}(-3)$$

$$= 2 + 7 + \frac{42}{2}(1) + \frac{210}{6}(0) + \frac{840}{24}(1) + \frac{2520}{120}(-1) + \frac{5040}{720}(2) + 1(-3)$$

$$= 2 + 7 + 21 + 0 + 35 - 21 + 14 - 3 = 55 = u_{10} .$$

The sum of consecutive terms of the Fibonacci series is given by:

$$\sum_{n=j}^{n=j+k-1} u_n = kd_{0,j} + \frac{k(k-1)}{2!} d_{1,j} + \frac{k(k-1)(k-2)}{3!} d_{2,j} + \cdots + \frac{k!}{k!} d_{k-1,j} \tag{6}$$

Example 2.

Calculate the sum of k = 9 consecutive Fibonacci numbers starting with u_j (j = 4).

$$\sum_{n=j}^{n=j+k-1} u_n = \sum_{n=4}^{n=9+4-1=12} u_n = 9(3) + \frac{9 \times 8}{2}(2) + \frac{9 \times 8 \times 7}{3 \times 2}(1) + \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2}(1)$$

$$+ \frac{9 \times 8 \times 7 \times 6 \times 5}{5 \times 4 \times 3 \times 2}(0) + \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4}{6 \times 5 \times 4 \times 3 \times 2}(1) + \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3}{7 \times 6 \times 5 \times 4 \times 3 \times 2}(-1)$$

$$+ \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}(2) + \frac{9!}{9!}(-3)$$

$$= 27 + 72 + 84 + 126 + 0 + 84 - 36 + 18 - 3 = 372$$

$$\sum_{n=4}^{n=12} u_n = 3 + 5 + 8 + 13 + 21 + 34 + 55 + 89 + 144 = 372$$

It is noted that $d_{k,j} \equiv d_{0,k-j}$ such that when $j - k = \pm p$, d_0 is the same numerically and is positive when p is positive or when p is negative and odd. However, d_0 is negative when p is negative and even.

Since d_0, the zero-order of difference, is the same as the Fibonacci series, u_n, equations 5 and 6 may be written in terms of u_n provided reference is made to the proper sign of u_{j-k} when $j - k$ is negative. Thus, examining the table of d_k and j forms, it may be seen that $d_{0,j} = u_j$, $d_{1,j} = d_{0,j-1} = u_{j-1}$, etc. Hence, equations 5 and 6 may be recast as follows:

$$(7)^\star \quad u_{k+j} = u_j + ku_{j-1} + \frac{k(k-1)}{2!}u_{j-2} + \frac{k(k-1)(k-2)}{3!}u_{j-3} + \cdots + \frac{k!}{k!}u_{j-k}$$

and

$$(8)^\star \quad \sum_{n=j}^{n=j+(k-1)} u_n = ku_j + \frac{k(k-1)}{2!}u_{j-1} + \frac{k(k-1)(k-2)}{3!}u_{j-2} + \cdots + \frac{k!}{k!}u_{j-(k-1)}$$

*Provided the sign of u_{j-k} is:

Positive when $j - k$ is positive

Positive when $j - k$ is negative and odd

Negative when $j - k$ is negative and even.

Also, $u_0 = 0$.

Example 3. Let $j = 3$ and $k = 7$. Calculate $u_{k+j} = u_{10}$

$$u_{10} = u_3 + 7u_2 + \frac{7 \times 6}{2}u_1 + \frac{7 \times 6 \times 5}{3 \times 2}u_0 + \frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2}u_{-1} + \frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2}u_{-2}$$

$$+ \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{6 \times 5 \times 4 \times 3 \times 2}u_{-3} + \frac{7!}{7!}u_{-4}$$

$$= 2 + 7(1) + 21(1) + 35(0) + 35(1) + 21(-1) + 7(2) + 1(-3)$$

$$= 2 + 7 + 21 + 0 + 35 - 21 + 14 - 3 = 55 = u_{10}$$

Example 4. Let $j = 3$ and $k = 7$. Calculate

$$\sum_{n=j}^{n=j+(k-1)} u_n = \sum_{n=3}^{n=3+7-1=9} u_n$$

$$\sum_{n=3}^{n=9} u_n = 7u_3 + \frac{7 \times 6}{2} u_2 + \frac{7 \times 6 \times 5}{3 \times 2} u_1 + \frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2} u_0 + \frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2} u_{-1}$$

$$+ \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{6 \times 5 \times 4 \times 3 \times 2} u_{-2} + \frac{7!}{7!} u_{-3}$$

$$= 7(2) + 21(1) + 35(1) + 35(0) + 21(1) + 7(-1) + 1(2)$$

$$= 14 + 21 + 35 + 0 + 21 - 7 + 2 = 86$$

$$= u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9$$

$$= 2 + 3 + 5 + 8 + 13 + 21 + 34 = 86$$

NB. The numbers in parentheses in Examples 3 and 4 are the numerical values appropriate for u_{j-k} with signs as provided above.

It is agreed that the above equations do not provide the least laborious way of calculating u_n or Σu_n but they do show that there is a relation between the Fibonacci series and the Binomial.

REFERENCES

1. Department of Defense Handbook H109, Statistical Procedures for Determining Validity of Suppliers' Attributes Inspection, 6 May 1960.
2. Dodge and Romig, Sampling Inspection Tables, Second Edition, 1959, John Wiley and Sons, Inc.
3. G. Chrystal, Textbook of Algebra, Second Edition, Chelsea Publishing House, Vol. II, p. 398.

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