A COMBINATORIAL PROBLEM INVOLVING FIBONACCI NUMBERS

J. L. Brown, Jr.
The Pennsylvania State University, University Park, Pa.

In Advanced Problem H-70 (this Quarterly, Vol. 3, No. 4, p. 299), C. A. Church, Jr. proposed the following combinatorial result:

"For $n = 2m$, show that the total number of k-combinations of the first n natural numbers such that no two elements i and $i + 2$ appear together in the same selection is F^2_{m+2} and if $n = 2m + 1$, the total is $F_{m+2}F_{m+3}$." (Solution appears in [1].)

The purpose of this note is to consider by a different method a more general combinatorial problem which includes Church's problem as a special case. As in the latter problem, the explicit solution will be seen to be expressible entirely in terms of Fibonacci numbers.

PROBLEM: Given the set S consisting of the first n positive integers and a fixed integer ν satisfying $0 < \nu \leq n$, how many different subsets A of S (including the empty subset) can be formed with the property that $a' - a'' \neq \nu$ for any two elements a', a'' of A (that is, subsets A such that integers i and $i + \nu$ do not both appear in A for any $i = 1, 2, \cdots, n - \nu$)?

Church's problem is then recovered from the above formulation on taking $\nu = 2$.

For the solution of the general problem, we let $n = m\ \ + r$ with m an integer and $0 \leq r \leq \nu$, so that $n = r \pmod{\nu}$. Each subset A of S can be made to correspond to an ordered binary sequence of n terms, $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, by the rule that $\alpha_i = 1$ if $i \in A$ and $\alpha_i = 0$ if $i \notin A$. For a given subset A and its corresponding binary sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, we define ν ordered binary sequences $A_1, A_2, \cdots, A_\nu$ as follows: For $1 \leq j \leq r$,

$$A_j = (\alpha_j, \alpha_{j+\nu}, \alpha_{j+2\nu}, \cdots, \alpha_{j+m\nu})$$

and for $r < j \leq \nu$

$$A_j = (\alpha_j, \alpha_{j+\nu}, \alpha_{j+2\nu}, \cdots, \alpha_{j+(m-1)\nu}) .$$

Note that each of the terms $\alpha_1, \alpha_2, \cdots, \alpha_n$ is included in one and only one of these sequences, since for $j = 1, 2, \cdots, \nu - 1$, the sequence A_j contains all α_i's with $i \equiv j \pmod{\nu}$ while A_ν contains all α_i's with $i \equiv 0 \pmod{\nu}$.

Now if the subset A under consideration satisfies the problem constraint, then clearly none of the sequences $\{A_j\}_1^\nu$ can contain two consecutive ones; conversely, if A contains both i and $i + \nu$ for some i_0 satisfying $1 \le i_0 \le n - \nu$, then the sequence A_k, where $k \equiv i_0 \pmod{\nu}$ will contain two successive ones. Thus the subset A under consideration will satisfy the given constraint if and only if each A_j $(j = 1, 2, \cdots, \nu)$ is a binary sequence without consecutive ones. But it is well known ([2], Problem 1(b), p. 14; [3], pp. 166-167) that the total number of binary sequences of length t without consecutive ones is F_{t+2}. Since each of the r sequences $A_1, A_2, \cdots, A_r$ has length m + 1 and each of the remaining $\nu - r$ sequences $A_{r+1}, \cdots, A_\nu$ has length m, it follows that the total number of subsets of A with the desired property is

$$F_{m+3}^{r} F_{m+2}^{\nu-r}$$

To obtain Church's result, we take $\nu = 2$ and let $n = 2m + r$ where $r = 0$ or $r = 1$, so that $n \equiv r \pmod 2$. Then the total number of k-combinations of the first n integers such that no elements i and i + 2 appear together is

$$F_{m+3}^{0} F_{m+2}^{2} = F_{m+2}^{2} \quad \text{if } r = 0 \text{ (n even)}$$

and

$$F_{m+3} F_{m+2} \quad \text{if } r = 1 \text{ (n odd)} .$$

Additional references dealing with the case $\nu = 2$ may be found in [1].

REFERENCES

1. C. A. Church, Jr., Problem H-70, Solution and Comments, The Fibonacci Quarterly, Vol. 5, No. 3, October, 1967, pp. 253-255.

2. J. Riordan, An Introduction to Combinatorial Analysis, John Wiley and Sons, Inc., N. Y. C., 1958. Problem 1(b), p. 14.

3. J. L. Brown, Jr., "Zeckendorf's Theorem and Some Applications," The Fibonacci Quarterly, Vol. 2, No. 3, Oct. 1964, pp. 163-168.

EDITORIAL NOTE: The restraint that $0 \le \nu \le n$ can be removed. Set m = 0, so that number of subsets becomes $F_{m+3}^{r} F_{m+2}^{\nu-r} = F_3^r F_2^{\nu-r} = 2^r$ as is well known for the numbers of subsets of 1, 2, 3, ..., n without constraints. V.E.H.

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