FORMULAS FOR DECOMPOSING F_{3n}/F_n, F_{5n}/F_n and L_{5n}/L_n INTO A SUM OR DIFFERENCE OF TWO SQUARES

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(1) $$F_{3n}/F_n = L_n^2 - (-1)^n$$

(1.1) $$F_{6n}/F_{2n} = L_{2n}^2 - 1 = (L_{2n} - 1)(L_{2n} + 1)$$

(1.2) $$F_{3(2n+1)}/F_{2n+1} = L_{2n+1}^2 + 1$$

(2) $$F_{5n}/F_n = (L_{2n} + (-1)^n)^2 - (-1)^n L_n^2$$

(2.1) $$F_{10n}/F_{2n} = (L_{4n} + 1)^2 - L_{2n}^2 = (L_{4n} + 1 - L_{2n})(L_{4n} + 1 + L_{2n})$$

(2.2) $$F_{5(2n+1)}/F_{2n+1} = (L_{4n+2} - 1)^2 + L_{2n+1}^2$$

(3) $$L_{5n}/L_n = (L_{2n} - (-1)^n 3)^2 + (5F_n)^2$$

(3.1) $$L_{10n}/L_{2n} = (L_{4n} - 3)^2 + (5F_n)^2$$

(3.2) $$L_{5(2n+1)}/L_{2n+1} = (L_{4n+2} - 3)^2 - (5F_{2n+1})^2 = (L_{4n+2} - 3 - 5F_{2n+1})(L_{4n+2} - 3 + 5F_{2n+1})$$

The formulas (1), (2), (3) can be easily verified by putting

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad \alpha\beta = -1,$$

and, for (3), also $\alpha - \beta = \sqrt{5}$.

Since for $n > 0$, (3.1) gives a decomposition of L_{10n}/L_{2n} into a sum of two squares, and since any divisor of a sum of two squares is $\equiv 1 \pmod 4$, it follows that any primitive divisor of L_{10n}, $n > 0$, is $\equiv 1 \pmod 4$.

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