where P_n is the Pell number defined by $P_1 = 1$, $P_2 = 2$, and $P_{n+2} = 2P_{n+1} + P_n$.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Letting $w = 2x \pm 1$ changes $x^2 + (x \pm 1)^2 = z^2$ into $w^2 - 2z^2 = -1$. Let Z be the ring of the integers and let $Z\sqrt{2}$ be the ring consisting of the real numbers $\alpha = z + b\sqrt{2}$ with a and b in Z. Let V consist of the positive real numbers $\alpha = a + b\sqrt{2}$ of $Z[\sqrt{2}]$ such that $a^2 - 2b^2 = -1$. Then V can be shown to be a group under multiplication. Since V has no number between 1 and $1 + \sqrt{2}$, it follows that V is the cyclic group generated by $1 + \sqrt{2}$. The odd powers $(1 + \sqrt{2})^{2n-1}$ lead to $a^2 - 2b^2 = -1$. Therefore the positive integral solutions of $w^2 - 2z^2 = -1$ are obtained by equating "rational" and "irrational" parts of $w_n + z_n\sqrt{2} = (1 + \sqrt{2})^{2n-1}$, i.e.,

$$w_n = \left[(1 + \sqrt{2})^{2n-1} + (1 - \sqrt{2})^{2n-1}\right]/2, \quad z_n = \left[(1 + \sqrt{2})^{2n-1} - (1 - \sqrt{2})^{2n-1}\right]/2\sqrt{2}.$$

The desired formulas then may be found using the analogue $P_n = \left[(1 + \sqrt{2})^n - (1 - \sqrt{2})^n\right]/2\sqrt{2}$ of one of the Binet formulas.

Also solved by A. C. Shannon and the proposer.

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(Continued from p. 176)

$$P_3(x) = 32 - 13x - 99x^2 - 32x^3 + 9x^4 + x^5$$
$$P_4(x) = 243 + 1181x - 1952x^2 - 1271x^3 + 257x^4 + 32x^5$$
$$P_5(x) = 3125 + 7768x - 15851x^2 - 9752x^3 + 1944x^4 + 243x^5$$

$$\sum_{n=0}^{\infty} F^6_{n+k}x^n = \frac{P_k(x)}{1 - 13x - 104x^2 + 260x^3 + 260x^4 - 104x^5 - 13x^6 + x^7}$$

$$k = 0, 1, 2, 3, 4, 5, 6$$

$$P_0(x) = x(1 - 12x - 53x^2 + 53x^3 + 12x^4 - x^5)$$
$$P_1(x) = 1 - 12x - 53x^2 + 53x^3 + 12x^4 - x^5$$
$$P_2(x) = 1 + 51x - 207x^2 - 248x^3 + 103x^4 + 13x^5 - x^6$$
$$P_3(x) = 64 - 103x - 508x^2 - 157x^3 + 117x^4 + 12x^5 - x^6$$
$$P_4(x) = 729 + 6148x - 16{,}797x^2 - 16{,}523x^3 + 6{,}668x^4 + 831x^5 - 64x^6$$
$$P_5(x) = 15{,}625 + 59{,}019x - 206{,}063x^2 - 182{,}872x^3 + 76{,}644x^4 + 9413x^5 - 729x^6$$
$$P_6(x) = 262{,}144 + 1{,}418{,}937x - 4{,}245{,}372x^2 - 3{,}985{,}856x^3 + 1{,}634{,}413x^4 + 202{,}396x^5 - 15{,}625x^6$$

(Continued on p. 166.)