ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

Editorial Note: Keep those problem proposals coming, Folks!

H-153 *Proposed by J. Ramanna, Government College, Mercara, India.*

Show that

(i)
$$4\sum_{0}^{n} F_{3k+1}F_{3k+2}\,(2F_{3k+1}^2 + F_{6k+3})(2F_{3k+2}^2 + F_{6k+3}) = F_{3n+3}^6$$

(ii)
$$16\sum_{0}^{n} F_{3k+1}\,F_{3k+2}\,F_{6k+3}\,(2F_{6k+3}^2 - F_{3k}^2\,F_{3k+3}^2) = F_{3n+3}^8\ .$$

Hence generalize (i) and (ii) for F_{3n+3}^{2r}.

H-154 *Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

Show that for m, n, p integers ≥ 0,

$$\sum_{i,j,k\geq 0} \binom{m+1}{j+k+1}\binom{n+1}{i+k+1}\binom{p+1}{i+j+1}$$
$$= \sum_{a=0}^{m}\sum_{b=0}^{n}\sum_{c=0}^{p}\binom{m-a+b}{b}\binom{n-b+c}{c}\binom{p-c+a}{a},$$

and generalize.

H-155 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.*

The Fibonacci polynomials are defined by

$$f_{n+1}(x) = x f_n(x) + f_{n-1}(x)$$

with $f_1(x) = 1$ and $f_2(x) = x$. Let $z_{r,s} = f_r(x)f_s(y)$. If $z_{r,s}$ satisfies the relation

$$z_{r+4,s+4} + a z_{r+3,s+3} + b z_{r+2,s+2} + c z_{r+1,s+1} + d z_{r,s} = 0 ,$$

show that

$$a = c = -xy, \quad b = -(x^2 + y^2 + 2) \quad \text{and} \quad d = 1.$$

H-156 *Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

Prove the identity

$$\sum_{n=0} \frac{q^{n^2} z^n}{(q)_n} \prod_{k=1}^{n} (1 - q^k) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q)_{2k}} z^{-k}$$

$$- \sum_{n=-\infty}^{\infty} q^{n(n+1)} z^n \sum_{k=0}^{\infty} \frac{q^{(k+1)^2}}{(q)_{2k+1}} z^{-k} ,$$

where

$$(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n) .$$

H-157 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.*

A set of polynomials $c_n(x)$, which appears in network theory is defined by

$$c_{n+1}(x) = (x + 2)c_n(x) - c_{n-1}(x) \qquad (n \geq 1)$$

with

$$c_0(x) = 1 \qquad \text{and} \qquad c_1(x) = (x + 2)/2 .$$

(a) Find a polynomial expression for $c_n(x)$.

(b) Show that

$$2c_n(x) = b_n(x) + b_{n-1}(x) = B_n(x) - B_{n-1}(x) ,$$

where $B_n(x)$ and $b_n(x)$ are the Morgan-Voyce polynomials as defined in the Fibonacci Quarterly, Vol. 5, No. 2, p. 167.

(c) Show that $2c_n^2(x) - c_{2n}(x) = 1$.

(d) If

$$Q = \begin{bmatrix} (x+2) & -1 \\ 1 & 0 \end{bmatrix} ,$$

show that

$$\begin{bmatrix} c_n & -c_{n-1} \\ c_{n-1} & -c_{n-2} \end{bmatrix} = \frac{1}{2}(Q^n - Q^{n-2}) \quad \text{for} \quad (n \geq 2) .$$

Hence deduce that $c_{n+1}c_{n-1} - c_n^2 = x(x + 4)/4$.

SOLUTIONS
AT LAST

H-98 *Proposed by George Ledin, Jr., San Francisco, California.*

If the sequence of integers is designated as J, the ring identity as I, and the quasi-inverse of J as F, then $(I - J)(I - F) = I$ should be satisfied. For further information see R. G. Buschman, "Quasi Inverses of Sequences," American Mathematical Monthly, Vol. 73, No. 4, III (1966), p. 134.

Find the quasi-inverse sequence of the integers (negative, positive, and zero).

Solution by the proposer.

The sequence $u_{n+2} = au_{n+1} + bu_n$ with initial conditions $u_0 \neq 1$, u_1, has the quasi-inverse

$$v_{n+2} = Av_{n+1} + Bv_n ,$$

where

$$A = a + u_1 / (1 - u_0), \; B = b/(1 - u_0)$$

with initial conditions

$$v_0 = -u_0 / (1 - u_0), \; v_1 = -u_1/(1-u_0)^2 .$$

Since the sequence of integers is defined by the recurrence relation

$$u_{n+2} = 2u_{n+1} - u_n$$

with initial conditions $u_0 = 0$, $u_1 = 1$, its quasi-inverse is then

$$v_{n+2} = 3v_{n+1} - v_n$$

with initial conditions $v_0 = 0$, $v_1 = -1$ which yields

$$0, -1, -3, -8, -21, -55, -144, -377, \cdots, -F_{2n}, \cdots .$$

SUM PRODUCT!

H-120 *Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.*

The Fibonacci polynomials are defined by

$$f_{n+1}(x) = x \cdot f_n(x) + f_{n-1}(x)$$
$$f_1(x) = 1, \quad f_2(x) = x .$$

If $z_r = f_r(x) \cdot f_r(y)$, then show that

(i) z_r satisfies the recurrence relation,

$$z_{n+4} = xy \cdot z_{n+3} - (x^2 + y^2 + 2)z_{n+2} - xy \cdot z_{n+1} + z_n = 0 .$$

(ii) $$(x + y)^2 \cdot \sum_1^n z_r = (z_{n+2} - z_{n-1}) - (xy - 1)(z_{n+1} - z_n) .$$

Solution by C.B.A. Peck, Ordnance Research Laboratory, State College, Pennsylvania.

(i)
$$\begin{aligned} z_{n+4} &= f_{n+4}(x)f_{n+4}(y) \\ &= (xf_{n+3}(x) + f_{n+2}(x))\,(yf_{n+3}(y) + f_{n+2}(y)) \\ &= xyz_{n+3} + xf_{n+3}(x)\, f_{n+2}(y) + yf_{n+3}(y)\, f_{n+2}(x) \neq z_{n+2} \\ &= xyz_{n+3} + (x^2 + y^2 + 2)\, z_{n+2} - z_{n+2} \\ &\quad + xf_{n+1}(x)\, f_{n+2}(y) + yf_{n+1}(y)\, f_{n+2}(x) \end{aligned}$$

so that

$$\begin{aligned} z_{n+4} - xyz_{n+3} - (x^2 + y^2 + 2)z_{n+2} &= -xyz_{n+1} - xf_{n+1}(x)\, f_n(y) \\ &- yf_{n+1}(y)f_n(x) - z_n + xyz_{n+1} + xf_{n+1}(x)f_n(y) + xyz_{n+1} \\ &+ yf_{n+1}(y)\, f_n(x) = xyz_{n+1} - z_n, \quad \text{as desired.} \end{aligned}$$

(ii) $n = 2$: by expansion,

$$(x + y)^2(1 + xy) = (x^3 + 2x)(y^3 + 2y) - 1 - (xy - 1)(\,(x^2 + 1)(y^2 + 1) - xy).$$

Thus for an inductive proof we need only to show the r.h. and l.h. increments equal. The r.h. one is

$$\begin{aligned} z_{n+2} - z_{n-1} - (xy - 1)(z_{n+1} - z_n) - z_{n+1} + z_{n-2} + (xy - 1)(z_n - z_{n-1}) \\ = z_{n+2} - xyz_{n+1} + 2(xy - 1)z_n - xyz_{n-1} + z_{n-2}, \end{aligned}$$

which by (i) is

$$(x^2 + y^2 + 2)z_n + 2(xy - 1)z_n = (x + y)^2 z_n ,$$

the l.h. one.

Also solved by the proposer, B. King, A. Shannon, L. Carlitz, and C. Bridger.

IN SUMMATION

H-121 *Proposed by H.H. Ferns, University of Victoria, Victoria, B.C., Canada.*

Prove the following identity.

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{F_k}{F_{m-k}}\right)^i F_{mi+\lambda} = \left(\frac{F_m}{F_{m-k}}\right)^n F_{nk+\lambda} - F_\lambda \qquad (m \neq k),$$

where F_n is the n^{th} Fibonacci number, m, λ are any integers or zero and k is an even integer or zero.

Write the form the identity takes if k is an odd integer.

Find an analogous identity involving Lucas numbers.

Solution by the proposer.

The following identities will be required.

$$\text{(1)} \qquad \alpha^k F_m - \alpha^m F_k = (-1)^k F_{m-k}$$

$$\text{(2)} \qquad \beta^k F_m - \beta^m F_k = (-1)^k F_{m-k},$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$ and $F_n = (\alpha^n - \beta^n)/\sqrt{5}$.

The proof of (1) follows. The proof of (2) is similar

$$\begin{aligned} \alpha^k F_m - \alpha^m F_k &= \alpha^k\left(\frac{\alpha^m - \beta^m}{\sqrt{5}}\right) - \alpha^m\left(\frac{\alpha^k - \beta^k}{\sqrt{5}}\right) \\ &= \frac{\alpha^{m+k} - \alpha^k\beta^m - \alpha^{m+k} + \alpha^m\beta^k}{\sqrt{5}} \\ &= \alpha^k\beta^k \frac{\alpha^{m-k} - \beta^{m-k}}{\sqrt{5}} \\ &= (-1)^k F_{m-k} \\ &= F_{m-k}, \end{aligned}$$

since k is even.

Identities (1) and (2) may be written as follows:

$$1 + \left(\frac{F_k}{F_{m-k}}\right)\alpha^m = \left(\frac{F_m}{F_{m-k}}\right)\alpha^k \qquad (m \neq k) \tag{3}$$

$$1 + \left(\frac{F_k}{F_{m-k}}\right)\beta^m = \left(\frac{F_m}{F_{m-k}}\right)\beta^k \qquad (m \neq k)\ . \tag{4}$$

Let

$$\mu = \frac{F_k}{F_{m-k}} \quad \text{and} \quad \nu = \frac{F_m}{F_{m-k}}\ .$$

From (3) and (4) we derive the following:

$$(1 + \mu\alpha^m)^n - (1 + \mu\beta^m)^n = \nu^n(\alpha^{nk} - \beta^{nk}) \tag{5}$$

$$(1 + \mu\alpha^m)^n + (1 + \mu\beta^m)^n = \nu^n(\alpha^{nk} + \beta^{nk})\ . \tag{6}$$

From (5) we get

$$\sum_{i=1}^{n}\binom{n}{i}\mu^i(\alpha^{mi} - \beta^{mi}) = \nu^n(\alpha^{nk} - \beta^{nk})$$

$$\sum_{i=1}^{n}\binom{n}{i}\mu^i F_{mi} = \nu^n F_{nk}\ . \tag{7}$$

If L_n denotes the n^{th} Lucas number then $L_n = \alpha^n + \beta^n$ and from (6) we obtain

$$\sum_{i=1}^{n}\binom{n}{i}\mu^i L_{mi} = \nu^n L_{nk} - 2\ . \tag{8}$$

We now add corresponding members of (7) and (8) and simplify the result by applying the identity

$$F_n + L_n = 2F_{n+1} .$$

This gives

$$\sum_{i=1}^{n} \binom{n}{i} \mu^i F_{mi+1} = \nu^n F_{nk+1} - 1 . \tag{9}$$

Adding corresponding members of (7) and (9) and applying the recursion formula

$$F_n + F_{n+1} = F_{n+2}$$

to the result yields

$$\sum_{i=1}^{n} \binom{n}{i} \mu^i F_{mi+2} = \nu^n F_{nk+2} - 1 . \tag{10}$$

Repeating the last operation on (8) and (9) and on each successive pair of identities derived in this manner we get

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{F_k}{F_{m-k}}\right)^i F_{mi+\lambda} = \left(\frac{F_m}{F_{m-k}}\right)^n F_{nk+\lambda} - F_\lambda \qquad (m \neq k)$$

If k is an odd integer this identity takes the form

$$\sum_{i=1}^{n} (-1)^i \binom{n}{i} \left(\frac{F_k}{F_{m-k}}\right)^i F_{mi+\lambda} = \left(\frac{-F_m}{F_{m-k}}\right)^n F_{nk+\lambda} - F_\lambda \qquad (m \neq k) .$$

Beginning with the two identities

$$\alpha^k L_m - \sqrt{5}\,\alpha^m F_k = (-1)^k L_{m-k}$$
$$\beta^k L_m + \sqrt{5}\,\beta^m F_k = (-1)^k L_{m-k}$$

and following the procedure adopted above we arrive at the identity

$$\sum_{i=1}^{n} \binom{n}{i} 5^{\left[\frac{i+1}{2}\right]} \left(\frac{F_k}{L_{m-k}}\right)^i \phi_m^i = \left(\frac{L_m}{L_{m-k}}\right)^n L_{nk+\lambda} - L_\lambda$$

where

$$\phi_m^i = \begin{cases} F_{mi+\lambda} & \text{if } i \text{ is odd} \\ L_{mi+\lambda} & \text{if } i \text{ is even} \end{cases}$$

and k is an even integer or zero. If k is an odd integer this identity takes the form

$$\sum_{i=1}^{n} \binom{n}{i} 5^{\left[\frac{i+1}{2}\right]} \left(\frac{F_k}{L_{m-k}}\right)^i (-\phi_m)^i = \left(\frac{-L_m}{L_{m-k}}\right)^n L_{nk+\lambda} - L_\lambda$$

<u>Examples</u>. If $\lambda = 0$, $m = 1$, $k = 2$ the first identity gives us the well-known formula

$$\sum_{i=1}^{n} \binom{n}{i} F_i = F_{2n} .$$

The same values for these parameters when substituted in the second identity gives the not-so-well-known formula

$$-\binom{n}{1} 5F_1 + \binom{n}{2} 5L_2 - \binom{n}{3} 5^2F_3 + \binom{n}{4} 5^2L_4 - \binom{n}{5} 5^3F_5 + \cdots = (-1)^n L_{2n} - 2.$$

Also solved by L. Carlitz, and A. Shannon.

STIRLING PERFORMANCE

H-123 *Proposed by D. Lind, University of Virginia, Charlottesville, Virginia*

Prove

$$F_n = \sum_{m=0}^{n} \sum_{k=0}^{m} \mathscr{S}_n^{(m)} S_m^{(k)} F_k ,$$

where $S_r^{(s)}$ and $\mathscr{S}_r^{(s)}$ are Stirling numbers of the first and second kinds, respectively, and F_n is the n^{th} Fibonacci number.

Solution by the proposer.

Stirling numbers are defined by

$$x(x-1)\cdots(x-m+1) = \sum_{m=0}^{n} S_n^{(m)} x^m$$

$$x^n = \sum_{m=0}^{n} \mathscr{S}_n^{(m)} x(x-1)\cdots(x-m+1) .$$

Letting $a = (1+\sqrt{5})/2$, $b = (1-\sqrt{5})/2$, we have

$$a^n = \sum_{m=0}^{n} \mathscr{S}_n^{(m)} a(a-1)\cdots(a-m+1)$$

(1)

$$= \sum_{m=0}^{n} \mathscr{S}_n^{(m)} \sum_{k=0}^{m} S_m^{(k)} a^k .$$

Similarly,

$$b^n = \sum_{m=0}^{n} \sum_{k=0}^{m} \mathscr{S}_n^{(m)} S_m^{(k)} b^k .$$

It follows

$$(a^n - b^n)/\sqrt{5} = \sum_{m=0}^{n} \sum_{k=0}^{m} \mathscr{S}_n^{(m)} S_m^{(k)} (a^k - b^k)/\sqrt{5} ,$$

which is the desired result.

[(1) may be found in Jordan's Calculus of Finite Differences, page 183.]

Also solved by David Zeitlin.

BINET?

H-124 *(Corrected). Proposed by J. A. H. Hunter, Toronto, Canada.*

Prove the following identity:

$$F^2_{m+n}L^2_{m+n} - F^2_mL^2_m = F_{2n}F_{2+(2m+n)} ,$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

Solution by Paul Smith, University of Victoria, Victoria, B.C., Canada.

A routine computation shows that:

$$F^2_{m+n}L^2_{m+n} - F^2_mL^2_m = \frac{[(\alpha^{m+n} - \beta^{m+n})(\alpha^{m+n} + \beta^{m+n})]^2 - [(\alpha^m - \beta^m)(\alpha^m + \beta^m)]^2}{(\alpha - \beta)^2}$$

$$= \frac{(\alpha^{4(m+n)} + \beta^{4(m+n)} - 1) - (\alpha^{4m} + \beta^{4m} - 1)}{(\alpha - \beta)^2}$$

$$= \frac{(\alpha^{4(m+n)} + \beta^{4(m+n)}) - \alpha^{2n}\beta^{2n}(\alpha^{4m} + \beta^{4n})}{(\alpha - \beta)^2}$$

$$= \frac{(\alpha^{2n} - \beta^{2n})}{(\alpha - \beta)} \cdot \frac{(\alpha^{2(2m+n)} - \beta^{2(2m+n)})}{(\alpha - \beta)}$$

$$= F_{2n}F_{2(2m+n)} .$$

(It is merely necessary to observe that $\alpha\beta = -1$.)

Also solved by C. Bridger, M. Bicknell, A. Shannon, C.B.A. Peck, J. Wessner, F. D. Parker, M. N. S. Swamy, and R. Whitney.

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