LINEAR RECURSION RELATIONS — LESSON FIVE
RECURSION RELATIONS OF HIGHER ORDER

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The considerations applied to linear recursion relations of the second order form a pattern for dealing with relations of higher order. Given a linear recursion relation of the k^{th} order:

$$T_{n+1} = a_1 T_n + a_2 T_{n-1} + \cdots + a_k T_{n-k+1} , \tag{1}$$

where the quantities a_i and T_i are real, there would be an auxiliary equation

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} \cdots - a_k = 0 , \tag{2}$$

for which there could be real and distinct roots, multiple real roots or complex roots conjugate in pairs. The major difficulty that arises in a relation of this type is the problem of determining the roots which ordinarily would be approximate in value.

As an example, consider one extension of the Fibonacci sequences, namely, adding the last three terms, or adding the last four terms, and so on. The recursion relations and corresponding auxiliary equations would be:

$$T_{n+1} = T_n + T_{n-1} + T_{n-2} \quad \text{and} \quad x^3 - x^2 - 1 = 0 \tag{3}$$

$$T_{n+1} = T_n + T_{n-1} + T_{n-2} + T_{n-3} \quad \text{and} \quad x^4 - x^3 - x^2 - x - 1 = 0 . \tag{4}$$

If we look at the general type of this equation:

$$x^k = x^{k-1} + x^{k-2} + x^{k-3} + \cdots + x + 1, \tag{5}$$

it appears that since

$$2^k - 1 = 2^{k-1} + 2^{k-2} + 2^{k-3} + \cdots + 2 + 1 , \tag{6}$$

there should be a root near 2. The following table gives an approximation to this root for various values of k.

k	Approximation to Root near 2
3	1.83928676
4	1.92756198
5	1.96594824
6	1.98358285
7	1.99196420
8	1.99603118
9	1.99802948

Approximations, such as these, to real or complex roots can be determined, but expressing T_n in terms of them does not seem very satisfying. Nevertheless, as will be seen in a subsequent lesson, such evaluations of roots of the auxiliary equation provide interesting information regarding the generated sequence.

MULTIPLE ROOTS

The case of multiple roots calls for additional consideration. If a polynomial equation

$$a_0 x^k + a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_k = 0 \tag{7}$$

has a root of multiplicity s, then (7) can be written:

$$(x - r)^s F(x) = 0 , \tag{8}$$

where $F(x)$ is a polynomial of degree $k - s$. Clearly, this equation and the equations formed by setting the first $s - 1$ derivatives equal to zero are all satisfied by r. This provides a clue for dealing with roots of any multiplicity when found in the auxiliary equation of a recursion relation. For concreteness, let us consider a root r of multiplicity 3.

Let the equation having this multiple root be:

$$x^3 - ax^2 - bx - c = 0 . \tag{9}$$

Multiply both sides of the equation by x^n to obtain:

$$x^{n+3} - ax^{n+2} - bx^{n+1} - cx^n = 0 . \tag{10}$$

Take the derivative and set the resulting polynomial equal to zero.

$$(n + 3)x^{n+2} - a(n + 2)x^{n+1} - b(n + 1)x^n - cnx^{n-1} = 0 . \tag{11}$$

Repeat this operation on (11).

$$(n + 3)(n + 2)x^{n+1} - a(n + 2)(n + 1)x^n - b(n + 1)nx^{n-1} - cn(n - 1)x^{n-2} = 0. \tag{12}$$

The multiple root r must satisfy the relations (10), (11), and (12) so that on replacing x by r and multiplying (11) by r and (12) by r^2 we have the following three recursion relations for r.

$$r^{n+3} = ar^{n+2} + br^{n+1} + cr^n , \tag{13}$$

$$(n + 3)r^{n+3} = a(n + 2)r^{n+2} + b(n + 1)r^{n+1} + cnr^n , \tag{14}$$

$$(n + 3)(n + 2)r^{n+3} = a(n + 2)(n + 1)r^{n+2} + b(n + 1)nr^{n+1} + cn(n - 1)r^n . \tag{15}$$

On the basis of these recursion relations the indicated expression for T_n is:

$$T_n = An(n - 1)r^n + Bnr^n + Cr^n . \tag{16}$$

We show first that this relation continues to hold for succeeding values of n if it is true for three consecutive values. For if

$$T_{n+1} = A(n + 1)nr^{n+1} + B(n + 1)r^{n+1} + Cr^{n+1} , \tag{17}$$

and

$$T_{n+2} = A(n + 2)(n + 1)r^{n+2} + B(n + 2)r^{n+2} + C r^{n+2} , \tag{18}$$

then

$$T_{n+3} = a T_{n+2} + b T_{n+1} + c T_n$$

is equal to:

$$T_{n+3} = A(n + 3)(n + 2)r^{n+3} + B(n + 3)r^{n+3} + C r^{n+3} \tag{19}$$

on the basis of relations (13), (14), and (15).

Given three initial values T_1, T_2, and T_3, the relations they should satisfy on the basis of (16) would be:

$$\begin{aligned} T_1 &= \qquad\qquad Br + Cr \\ T_2 &= 2Ar^2 + 2Br^2 + Cr^2 \\ T_3 &= 6Ar^3 + 3Br^3 + Cr^3 \quad . \end{aligned} \tag{20}$$

The determinant of the coefficients of the unknowns A, B, C has a value of $-2r^6$, so that if r is not zero, there are unique solutions for A, B, and C. Thus three initial values T_1, T_2 and T_3 can be expressed in the form given by (16). It follows that this form will continue to hold for all values of n.

It may be noted in passing that if the multiple root has a value of 1, T_n reduces to a polynomial in n.

Example. Express the terms of the recursion relation

$$T_{n+1} = 7T_n - 17T_{n-1} + 14T_{n-2} + 4T_{n-3} - 8T_{n-4}$$

in terms of the roots of the auxiliary equation:

$$x^5 - 7x^4 + 17x^3 - 14x^2 - 4x + 8 = 0.$$

By synthetic division three equal roots, 2, are found and the residual quadratic has the roots

$$\frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \frac{1 - \sqrt{5}}{2} \quad .$$

Accordingly,

$$T_{n+1} = An(n - 1)x2^n + Bnx2^n + Cx2^n + Dr^n + Es^n$$

where

$$r = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad s = \frac{1 - \sqrt{5}}{2}$$

Example. For the recursion relation

$$T_{n+1} = 3T_n - 3T_{n-1} + T_{n-2} \;,$$

with initial values $T_1 = 5$, $T_2 = 8$, $T_3 = 17$, express T_n in terms of the roots of the auxiliary equation.

This equation is

$$x^3 - 3x^2 + 3x - 1 = 0 \;,$$

which has a triple root of 1. Thus

$$T_{n+1} = An(n - 1) + Bn + C \;,$$

a polynomial in n. Then

$$\begin{aligned} 5 &= B + C \\ 8 &= 2A + 2B + C \\ 17 &= 6A + 3B + C \quad , \end{aligned}$$

leading to the values $A = 3$, $B = -3$, $C = 8$, so that

$$T_{n+1} = 3n^2 - 6n + 8 .$$

PROBLEMS

1. Find the recursion relation satisfied by

$$T_n = 3n^2 - 5n + 4 + 2 \times 5^n .$$

2. Given the recursion relation

$$T_{n+1} = 6T_n - 11T_{n-1} + 6T_{n-2} ,$$

and initial values

$$T_1 = 8, \qquad T_2 = 15, \qquad T_3 = 22 .$$

Express the general term T_n in terms of the roots of the auxiliary equation.

3. S_n is the Fibonacci sequence $3, 7, 10, 17, 27, \cdots$, and R_n is the geometric progression $5, 15, 45, 135, \cdots$

$$T_n = R_n + S_n .$$

Find the recursion relation for T_n.

4. If $T_n = 3n + 2 + 2(-1)^n + F_n$, find the recursion relation for T_n.

5. If $T_1 = 13$, $T_2 = 15$, $T_3 = 22$ and $T_{n+1} = 4T_n - T_{n-1} - 2T_{n-2}$ express T_n in terms of the roots of the auxiliary equation of this recursion relation.

(Solutions are on p. 302.)

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NOTICE