ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contribution are asked to enclose self-addressed stamped postcards.

B-166 Suggested by David Zeitlin's solutions to B-148, 149, and 150.

Let a and b be distinct numbers, $U_n = (a^n - b^n)/(a - b)$, and $V_n = a^n + b^n$. Establish generalizations of the formulas

(a) $$F_{(2^t n)} = F_n L_n L_{2n} \cdots L_{(2^{t-1} n)}$$

(b) $$L_{n+1} L_{n+3} + 4(-1)^{n+1} = 5F_n F_{n+4}$$

of B-148 and B-149 in which one deals with U_n and V_n instead of F_n and L_n.

B-167 Proposed by A. G. Shannon, University of Papua and New Guinea, Boroko, T. P. N. G.

Let L_n be the n^{th} Lucas number defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$. For which values of n is

$$nL_{n+1} > (n + 1)L_n \quad ?$$

B-168 Proposed by S. H. L. King, Jacksonville University, Jacksonville, Florida.

Using each of six of the nine positive digits $1, 2, \cdots, 9$ exactly once, form an integer z such that each of z, 2z, 3z, 4z, 5z, and 6z contains the same six digits once and once only.

B-169 Proposed by C. C. Yalavigi, Government College, Mercara, India.

Prove the following identities:

(a) $$F_n^4 + F_{n-1}^4 + F_{n+1}^4 = 2(F_nF_{n-1} - F_{n+1})^2$$

(b) $$F_n^5 + F_{n-1}^5 - F_{n+1}^5 = 5F_nF_{n-1}F_{n+1}(F_nF_{n-1} - F_{n+1}^2)$$

where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Show that these are two cases of an infinite sequence of identities.

B-170 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let the binomial coefficient $\binom{m}{r}$ be zero when $m < r$ and let

$$S_n = \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} .$$

Show that $S_{n+2} - S_{n+1} + S_n = 0$ and hence $S_{n+3} = -S_n$ for $n = 0, 1, 2, \cdots$.

B-171 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $\binom{m}{r} = 0$ for $m < r$ and let

$$T_n = \sum_{j=0}^{\infty} \binom{n-2j}{2j} .$$

Obtain a fourth-order homogeneous linear recurrence formula for T_n.

SOLUTIONS

CORRECTION. In the solution to B-128 in Vol. 6, No. 4 (Oct. 1968), line 2 from the bottom of p. 295 should read:

$$S_{4n} = f_{4n+2} - f_2 = (F_{4n+1} - 1)f_2 + F_{4n}f_1 ,$$

and line 5 from the top of p. 296 should read:

$$F_{4n+1} - 1 = F_{2n}L_{2n+1} .$$

COMMENT. Mr. J. D. E. Konhauser, Macalester College, St. Paul, Minnesota, sent in the following on B-130a:

The five-disk problem is discussed in Mathematical Recreations and Essays by W. W. R. Ball, revised by H. S. M. Coxeter in 1938, on pages 97-99. Included is a reference to a 1915 paper by E. H. Neville in the Proceedings of the London Mathematical Society, second series, Vol. xiv, pp. 308-326.

ADDITIONS TO LISTS OF SOLVERS: Problem B-143 was also solved by D. V. Jaiswal (Indore, India), Amanda Neel, and A. G. Shannon (Boroko, T. P. N. G.) Problem B-143 was also solved by D. V. Jaiswal and A. G. Shannon. Problem B-146 was also solved by D. V. Jaiswal and A. G. Shannon.

TELESCOPING PRODUCT

B-148 Proposed by David Englund, Rockford College, Rockford, Illinois, and Malcolm Tallman, Brooklyn, New York.

Let F_n and L_n denote the Fibonacci and Lucas numbers and show that

$$F_{(2^t n)} = F_n L_n L_{2n} L_{4n} \cdots L_{(2^{t-1} n)} .$$

Solution by Douglas Lind, Cambridge University, Cambridge, England.

By the well-known formula $F_{2n} = F_n L_n$, we have

$$F_{2^t n} = F_{2^{t-1} n} L_{2^{t-1} n} = F_{2^{t-2} n} L_{2^{t-2} n} L_{2^{t-1} n} = \cdots = F_n L_n L_{2n} \cdots L_{2^{t-1} n} .$$

Also solved by Christine Anderson, Serge Hamelin (Canada), Bruce W. King, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, Michael Yoder, David Zeitlin, and the proposer.

A QUADRATIC IDENTITY

B-149 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that

$$L_{n+1}L_{n+3} + 4(-1)^{n+1} = 5F_nF_{n+4}.$$

Solution by Carol A. Vespe, Student, University of New Mexico, Albuquerque, New Mexico.

Let $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$. Since both sides of the equation are of the form

$$c_1(a^2)^n + c_2(ab)^n + c_3(b^2)^n ,$$

with constant c_i, it suffices to note that the identity holds for $n = 0, 1,$ and 2.

Also solved by Clyde A. Bridger, Juliette Davenport, Herta T. Freitag, Serge Hamelin (Canada), Bruce W. King, H. V. Krishna (Manipal, India), Douglas Lind, John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), C. C. Yalavigi (Mercara, India), Michael Yoder, David Zeitlin, and the Proposer.

ANOTHER QUADRATIC IDENTITY

B-150 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that

$$L_n^2 - F_n^2 = 4F_{n-1}F_{n+1}.$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Let U_n and V_n be solutions of $W_{n+2} = aW_{n+1} + bW_n$, where $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, and $V_1 = a$. Noting that

$$V_n = 2U_{n+1} - aU_n \equiv U_{n+1} + bU_{n-1},$$

we obtain

(1) $$V_n^2 - a^2U_n^2 = 4bU_{n-1}U_{n+1}.$$

The desired result is obtained from (1) with $a = b = 1$, $V_n \equiv L_n$, and $U_n \equiv F_n$.

Also solved by Clyde A. Bridger, Juliette Davenport, David Englund, Herta T. Freitag, Serge Hamelin (Canada), John E. Homer, Jr., Bruce W. King, H. V. Krishna (Manipal, India), Douglas Lind (England), John W. Milsom, C. B. A. Peck, Gerald Satlow, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, Michael Yoder, and the Proposer.

MISSING TERMS

B-151 Proposed by Hal Leonard, San Jose State College, San Jose, California.

Let $m = L_1 + L_2 + \cdots + L_n$ be the sum of the first n Lucas numbers. Let

$$P_n(x) = \prod_{i=1}^{n} (1 + x^{L_i}) = a_0 + a_1 x + \cdots + a_m x^m .$$

Let q_n be the number of integers k such that both $0 < k < m$ and $a_k = 0$. Find a recurrence relation for the q_n.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Note that

$$m = m_n = L_1 + L_2 + \cdots + L_n = L_{n+2} - 3$$

and that q_n is the number of integers in $\{1,2,3,\cdots,m-1\}$ that are not expressible in the form

$$c_1 L_1 + c_2 L_2 + \cdots + c_n L_n \;; \quad c_i \in \{0,1\} \quad \text{for} \quad 1 \leq i \leq n .$$

A generalization of this problem is dealt with by David A. Klarner in "Representations of N as a Sum of Distinct Elements from Special Sequences," Fibonacci Quarterly, Vol. 4, No. 4 (Dec. 1966), pp. 289-306.

Using

$$m_{2k} = L_1 + L_2 + L_3 + \cdots + L_{2k} = L_3 + L_5 + L_7 + \cdots + L_{2k+1}$$

$$m_{2k-1} = L_1 + L_2 + \cdots + L_{2k-1} = L_1 + (L_4 + L_6 + \cdots + L_{2k}) ,$$

and formula (43) on page 303 of Klarner's paper, one has

$$q_{2k} = m_{2k} - (F_4 + F_6 + \cdots + F_{2k+2}) = L_{2k+2} - 3 - (F_{2k+3} - F_3)$$

$$q_{2k-1} = m_{2k-1} - F_2 - (F_5 + F_7 + \cdots + F_{2k+1}) = L_{2k+1} - 3 - 1 - (F_{2k+2} - F_4).$$

Now $L_n = F_{n+1} + F_{n-1}$ leads to $q_n = F_{n+1} - 1$ for all n. Hence

$$(q_{n+2} + 1) = (q_{n+1} + 1) + (q_n + 1) \quad \text{or} \quad q_{n+2} = q_{n+1} + q_n + 1$$

Also solved by Serge Hamelin (Quebec, Canada), C. B. A. Peck, and Carol A. Vespe. Hamelin gave the homogeneous recursion formula $q_{n+3} = 2q_{n+2} - q_n$.

FIBONACCI ADDITION FORMULA

B-152 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Prove that

$$F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1} .$$

Solution by John E. Homer, Jr., Lisle, Illinois.

From the well-known formulas

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_m F_n$$

$$F_{m+n-1} = F_m F_n + F_{m-1}F_{n-1} ,$$

we have

$$F_{m+1}F_{n+1} - F_{m-1}F_{n-1} = F_{m+n+1} - F_{m+n-1} = F_{m+n}.$$

Also solved by Clyde A. Bridger, Juliette Davenport, David Englund, Herta T. Freitag, Serge Hamelin (Canada), Bruce W. King, H. V. Krishna (Manipal, India), Douglas Lind (England), John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, C. C. Yalavigi (Mercara, India), Michael Yoder, and the Proposer.

[Continued in p. 276.]

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