ON THE GROWTH OF $d_k(n)$

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1.) Let $d(n)$ denote the number of divisors of n, $\log_k n$ the k-fold iterated logarithm. It was shown by Wigert [1] that ($\exp z = e^z$)

$$d(n) < \exp\left((1+\epsilon)\log 2 \frac{\log n}{\log\log n}\right)$$

for all positive values of ϵ and all sufficiently large values of n, and that

$$d(n) > \exp\left((1-\epsilon)\log 2 \frac{\log n}{\log\log n}\right)$$

for an infinity of values of n.

Let $d_k(n)$ denote the k-fold iterated $d(n)$ (i.e.,

$$d_1(n) = d(n), \ (d_k(n) = d(d_{k-1}(n)), \ k \geq 2).$$

S. Ramanujan remarked in his paper [2] that

$$d_2(n) > 4^{\frac{\sqrt{2\log n}}{\log\log n}},$$

and that

$$d_3(n) > (\log n)^{\log\log\log\log n}$$

for an infinity of values of n.

Let ℓ_k denote the k^{th} element of the Fibonacci sequence (i.e.,

$$\ell_{-1} = 0, \ \ell_0 = 1, \ \ell_k = \ell_{k-1} + \ell_{k-2} \text{ for } k \geq 1).$$

We prove the following:

Theorem 1. We have

$$(1.1)\qquad d_k(n) < \exp\ (\log n)^{\frac{1}{\ell_k}+\epsilon}$$

for all fixed k, all positive ϵ and all sufficiently large values of n, further for every $\epsilon > 0$

$$(1.2)\qquad d_k(n) > \exp\left((\log n)^{\frac{1}{\ell_k}-\epsilon}\right)$$

for an infinity of values of n.

It is obvious that $d(n) < n$, if $n > 2$. For a general $n > 1$, let k(n) denote the smallest k for which $d_k(n) = 2$. We shall prove

Theorem 2.

$$(1.3)\qquad 0 < \limsup \frac{K(n)}{\log\log\log n} < \infty\ .$$

2.) The letters c, c_1, $c_2, \cdots$ denote positive constants, not the same in every occurrence. The p_i's denote the i^{th} prime number.

3.) First, we prove (1.2). Let r be large. Put $N_1 = 2\cdot 3 \cdots p_r$, where the p's are the consecutive primes. We define $N_2, \cdots, N_k$ by induction. Assume

$$(3.1)\qquad N_j = \prod_{i=1}^{S_j} p_i^{r_2},$$

then

$$(3.2)\qquad N_{j+1} = \left(p_1 \cdots p_{r_1}\right)^{p_1-1}\left(p_{r_1+1} \cdots p_{r_1+r_2}\right)^{p_2-1} \cdots \left(p_{r_1+\cdots+r_{S_j-1}+1} \cdots p_{r_1+\cdots+r_{S_j}}\right)^{p_{S_j}-1}$$

From (3.2) $d(N_{j+1}) = N_j$, and thus

$$d_k(N_k) = 2^r . \tag{3.3}$$

Let S_j and Γ_j denote the number of different and all prime factors of N_j, respectively. We have

$$S_1 = \Gamma_1 = r, \quad S_{j+1} = \Gamma_j . \tag{3.4}$$

Furthermore

$$S_{j+2} = \Gamma_{j+1} = \sum_{\nu=1}^{S_j} \gamma_\nu (p_\nu - 1) < p_{S_j} \sum_{\nu=1}^{S_j} \gamma_\nu < c\,\Gamma_j S_j \log S_j , \tag{3.5}$$

since $p_\ell < c_\ell \log \ell$ for $\ell \geq 2$. Hence by (3.4)

$$S_{j+2} < c\, S_{j+1} S_j \log S_j \qquad (j \geq 1) , \tag{3.6}$$

follows.

Using the elementary fact that

$$\sum_{i=1}^{\ell} \log p_i < cp_\ell < c\ell \log \ell ,$$

we obtain from (3.2),

$$\log N_{j+1} \leq p_{S_j} \sum_{i=1}^{\Gamma_j} \log p_i \leq c\, S_j \Gamma_j (\log \Gamma_j)^2 = cS_j S_{j+1} (\log S_{j+1})^2 . \tag{3.7}$$

From (3.3), (3.4) we easily deduce by induction that for every $\epsilon > 0$ and sufficiently large r

$$S_1 = r, \quad \Gamma_1 = r, \quad S_2 = r, \quad \Gamma_2 < r^{2+\epsilon}, \; S_3 < r^{2+\epsilon}, \; \Gamma_3 < r^{3+\epsilon}, \cdots ,$$

$$S_k < r^{\ell_{k-1}+\epsilon} , \; \Gamma_k \leq r^{\ell_k+\epsilon} .$$

Using (3.7), we obtain that

$$\log N_k \le r^{\ell_k + \epsilon},$$

whence

$$d_k(N_k) = 2^r \ge \exp\left((\log N_k)^{1/\ell_k - \epsilon} \right),$$

which proves (1.2).

4.) Now we prove (1.1). Let $N_0, N_1, \cdots, N_k$ be an arbitrary sequence of natural numbers, such that

$$d(N_{j+1}) = N_j ,$$

for $j = 0, 1, \cdots, k - 1$.

Let B denote an arbitrary quantity in the interval

$$(\log \log N_k)^{-c} \le B \le (\log \log N_k)^{c} ,$$

not necessarily the same at every occurrence.

We prove

$$\log N_k \ge B(\log N_0)^{\ell_k} , \tag{4.1}$$

whence (1.1) immediately follows.

In the proof of (4.1) we may assume that $\log N_0 \ge (\log N_k)$, with a positive constant $\delta < 1/\ell_k$.

Let

$$N_1 = \prod_{i=4}^{S_1} q_i^{\alpha_i - 1} .$$

Then

$$N_0 = \prod_{i=1}^{S_1} \alpha_i .$$

Since

$$2^{\alpha_i - 1} \le q_i^{\alpha_i - 1} \,\Big|\, N_1 ,$$

we have

$$\alpha_i \le c \log N_1 .$$

Hence

$$(\log 2)S_1 \le \log N_0 = \Sigma \log \alpha_i \le (\log\log N_1 + c)S_1 ,$$

i.e.,

$$\log N_0 = BS_1 .$$

We need the following:

Lemma. Suppose that for some integer j, $1 \le j \le k - 1$,

$$Q_1^{\gamma_1 - 1} \cdots Q_A^{\gamma_A - 1} \,\Big|\, N_j , \tag{4.2}$$

where $Q_1, \cdots, Q_A$ are different prime numbers and

$$A \ge BS_1^{\ell_{j-1}} ; \quad Q_i \ge BS_1^{\ell_{j-1}} , \quad \gamma_i \ge BS_1^{\ell_{j-2}} \quad (i = 1, \cdots, A) . \tag{4.3}$$

Then either

$$\log N_{j+1} \ge (\log N_0)^{\ell_k} , \tag{4.4}$$

or

$$r_1^{\beta_1 - 1} \cdots r_C^{\beta_C - 1} \,\Big|\, N_{j+1} , \tag{4.5}$$

where $r_1, \cdots, r_C$ are different primes and

$$C \ge BS_1^{\ell_j}, \quad r_i \ge BS_1^{\ell_j}, \quad \beta_i \ge BS_1^{\ell_{j-1}} \quad (i = 1, \cdots, C) . \tag{4.6}$$

To prove the lemma, let

$$N_{j+1} = \prod^{S_{j+1}} t_i^{\delta_i - 1} , \quad t_i \text{ primes} .$$

Since $d(N_{j+1}) = N_j$, by (4.2),

$$\prod_{i=1}^{A} Q_i^{\gamma_i - 1} \Bigg| \prod^{S_{j+1}} \delta_i = N_j \ . \tag{4.7}$$

Assume first that there is a δ_i which has at least $2\ell_k$ (not necessarily distinct) prime divisors amongst the Q_i. We then have

$$\log N_{j+1} \geq \frac{1}{2}\delta_i \log t_i \geq \frac{\log 2}{2}\ \delta_i \geq \left(BS_1^{\ell_{j-1}}\right)^{2\ell_k} \geq$$
$$\geq (BS_1)^{2\ell_k} \geq (\log N_0)^{\ell_k} \ ,$$

if N_0 is sufficiently large, i.e. (4.4) holds. Then by (4.2), the number D of δ's, each of which contains a prime divisor amongst the Q's satisfies the inequality

$$D \geq \frac{1}{2\ell_k} \sum_{i=1}^{A} (\gamma_i - 1) \geq \frac{A}{4\ell_k} \min \gamma_2 \geq ABS_1^{\ell_{j-2}} \geq BS_1^{\ell_{j-2}+\ell_{j-1}} = BS_1^{\ell_j} \ . \tag{4.8}$$

Without loss of generality, we assume that these δ's are $\delta_1, \cdots, \delta_D$ and $t_1 > t_2 > \cdots > t_D$ in (4.7). Since at least one Q divides $\delta_i (i \leq D)$, by (4.3), we have

$$\delta_i > BS_1^{\ell_{j-1}} \ .$$

Furthermore it is obvious that $t_{[D/2]} > D$. By choosing

$$C = D - \frac{D}{2} \ , \quad r_i = t_i, \quad \beta_i = \delta_i \quad (i = 1, \cdots, C) \ ,$$

we obtain (4.5) and (4.6).

This completes the proof of the Lemma.

Now (4.1) rapidly follows. Indeed, the validity of (4.4) for some j, $1 \leq j \leq k - 1$, immediately implies (4.1). So we may assume that (4.4) does not hold for $j = 1, \cdots, k - 1$. Now we use the Lemma for $j = 1, \cdots, k - 1$. Since N_1 has S_1 different prime divisors ($[1/2\ S_1]$ of these is greater than S_1)

the conditions (4.2), (4.3) are satisfied for $j = 1$. Hence (4.5)-(4.6) holds, i.e., the conditions (4.2)-(4.3) hold for $j = 2$. By induction we obtain that N_k has at least

$$BS_1^{\ell^{k-1}}$$

distinct prime factors each with the exponent greater than $BS_1^{\ell^{k-2}}$. Let

$$N_k = \Pi P_i^{\rho_i - 1} .$$

Since

$$\log N_k > \frac{1}{4} \sum \rho_i ,$$

we have

$$\log N_k > BS_1^{\ell^{k-1}+\ell^{k-2}} = B(\log N_0)^{\ell^k} .$$

Consequently (4.1) holds.

5.) Proof of Theorem 2. Using (1.1) in the form

$$d_2(n) < \exp\left((\log n)^{2/3}\right)$$

for $n \geq c$, and applying this k times, we have

$$\log d_{2k}(n) < (\log n)^{(2/3)^k}, \text{ when } d_{2k-2}(n) \geq c. \tag{5.1}$$

Equation (5.1) implies the upper bound in (1.3) by a simple computation.

For the proof of the lower bound we use the construction as in 3). Let r be so large that

$$cS_{j+1} (\log S_{j+1})^2 < S_{j+1}^{1+\epsilon}$$

in (3.6). Using that

$$\log N_{j+1} \leq (\log N_j)^{2+\epsilon} .$$

Thus

$$\log N_k \leq (\log N_1)^{(2+\epsilon)^k} ,$$

hence by taking logarithms twice,

$$K(N_k) \geq k \geq c_1 \log_3 N_k ,$$

which completes the proof of (1.3).

Denote by $L(n)$ the smallest integer for which $\log n_{L(n)} < 1$. We conjecture that

$$\frac{1}{n} \sum_{m=1}^{n} K(m)$$

increases about like $L(n)$, but we have not been able to prove this.

REFERENCES

1. Wigert, Sur l'ordre de grandeur du nombre des diviseurs d'un entier, Arkiv för Math. 3(18), 1-9.
2. S. Ramanujan, "Highly Composite Numbers," Proc. London Math. Soc., 2(194), 1915, 347-409, see p. 409.

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CORRECTION

On p. 113 of Volume 7, No. 2, April, 1969, please make the following changes:

Change the author's name to read George E. Andrews. Also, change the name "Einstein," fourth line from the bottom of p. 113, to "Eisenstein."

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