NOTE ON THE INITIAL DIGIT PROBLEM

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The initial digit problem is concerned with the distribution of the first digits which occur in the set of all positive integers. There are many possible interpretations of the heuristic question, "What is the probability that an integer chosen at random has initial digit equal to a?" [1]. If $A = \{a_n\}$ is the set of all positive integers with initial digit a, then the asymptotic density [2] of A would provide a suitable answer to this question if it exists. However, it is easily shown that the asymptotic density doesn't exist.

The purpose of this note is to show that the logarithmic density [2] of A exists and is equal to $\log(1 + 1/a)$, where $\log x$ is the common logarithm. This result is in agreement with previous solutions of the initial digit problem [1]. It is also of interest to note that the logarithmic density exists and is equal to the asymptotic density whenever the latter exists [2].

The logarithmic density $\delta(A)$ is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{\ln n} \sum_{a_\nu \le n} \frac{1}{a_\nu} ,$$

and the lower and upper logarithmic densities $\underline{\delta}(A)$ and $\overline{\delta}(A)$ are obtained by replacing lim by $\underline{\lim}$ and $\overline{\lim}$ respectively. Now it is obvious that

$$\underline{\delta}(A) = \lim_{k\to\infty} \frac{1}{\ln(a\,10^k - 1)} \sum_{a_\nu \le a10^k - 1} \frac{1}{a_\nu}$$

$$= \lim_{k\to\infty} \frac{1}{k \ln 10} \sum_{\nu=1}^{k-1} [H((a+1)10^\nu - 1) - H(a\,10^\nu - 1)]$$

$$= \lim_{k\to\infty} \frac{1}{k \ln 10} \sum_{\nu=1}^{k-1} [H((a+1)10^\nu) - H(a\,10^\nu)] ,$$

where

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} .$$

Using the well-known asymptotic formula [3] $H(n) = \ln n + \gamma + 0(1/n)$, we get

$$\begin{aligned}\underline{\delta}(A) &= \lim_{k\to\infty} (k - 1)(\ln(a + 1) - \ln a)/k \ln 10 \\ &= \ln(1 + 1/a)/\ln 10 = \log(1 + 1/a) .\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{\delta}(A) &= \lim_{k\to\infty} \frac{1}{\ln((a + 1)10^k - 1} \sum_{a_\nu \le (a+1)10^k - 1} \frac{1}{a_\nu} \\ &= \lim_{k\to\infty} \frac{1}{k \ln 10} \sum_{\nu=1}^{k} [H((a + 1) 10^\nu) - H(a 10^\nu)] \\ &= \log(1 + 1/a) = \underline{\delta}(A) ,\end{aligned}$$

and the desired result follows.

REFERENCES

1. B. J. Flehinger, "On the Probability that a Random Integer has Initial Digit A," Amer. Math. Monthly, 73 (1966), 1056-1061.
2. H. Halberstam and K. F. Roth, Sequences, Vol. 1, Oxford, 1966.
3. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1960.

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