AN APPLICATION OF THE LUCAS TRIANGLE

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1. INTRODUCTION

Consider the integer triangle whose entries are given by

$$A_{j,0} = 1, \quad A_{j,j} = 2, \quad j = 1, 2, 3, \cdots ;$$

$$A_{n+1,j} = A_{n,j} + A_{n,j-1} \quad (0 < j < n, \ n \geq 1).$$

The first few lines of the triangle are listed left-justified below:

A:

1	2					
1	3	2				
1	4	5	2			
1	5	9	7	2		
1	6	14	16	9	2	
1	7	20	30	25	11	2

One notes that the recurrence relation is the same as the one for Pascal's triangle. Apart from no $A_{0,0}$ term the array is really the sum of two Pascal triangles. The rising diagonal sums are the Lucas numbers, $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$. The $A_{0,0} = 2$ would also add $L_0 = 2$ to the rising diagonal sum sequence. The triangular array is now the Lucas triangle of Mark Feinberg [1]. It is also closely related to a convolution triangle [3].

Consider the new array obtained in a simple way from our first array A by shifting the j^{th} column down j places $(j = 1, 2, 3, \cdots)$. The column on the left is the 0^{th} column.

B:

1				
1	2			
1	3			
1	4	2		
1	5	5		
1	6	9	2	
1	7	14	7	
1	8	20	16	2
1	...	...	...	...

The relationship is

$$B_{i,j} = A_{i-j,j} \qquad 0 \le j \le [i/2] \quad ,$$

where [x] is the greatest integer not exceeding x. The recurrence relation for $B_{i,j}$ is

$$B_{i,o} = 1 \qquad \text{for all } i \quad ,$$

$$B_{i,j} = B_{i-1,j} + B_{i-2,j-1}, \qquad 1 \le j \le [i/2]$$

along with other useful relations true for all j:

$$\begin{aligned} B_{2j,j} &= 2 \\ B_{2j+1,j} &= 2j + 1 \\ B_{2j+1,j+1} &= 0 \quad . \end{aligned}$$

2. ANOTHER ARRAY

Harlan Umansky [2] laid out the following display of formulas for powers of Lucas numbers.

$$
\text{C:}\quad
\begin{aligned}
L_n^1 &= L_n \\
L_n^2 &= L_{2n} + 2(-1)^n \\
L_n^3 &= L_{3n} + 3(-1)^n L_n \\
L_n^4 &= L_{4n} + 4(-1)^n L_n^2 - 2 \\
L_n^5 &= L_{5n} + 5(-1)^n L_n^3 - 5L_n \\
L_n^6 &= L_{6n} + 6(-1)^n L_n^4 - 9L_n^2 + 2(-1)^n \\
L_n^7 &= L_{7n} + 7(-1)^n L_n^5 - 14L_n^3 + 7(-1)^n L_n \\
L_n^8 &= L_{8n} + 8(-1)^n L_n^6 - 20L_n^4 + 16(-1)^n L_n^2 - 2
\end{aligned}
$$

The display given in [2] contains 7 missing pairs of parentheses. The above displayed form was suggested by Edgar Karst who, along with Brother Alfred Brousseau, noted the typing errors in [2]. Surely, we note that exclusive of signs, the coefficients in display C are precisely those of Array B. We shall prove the theorem:

Theorem 1.

$$L_n^m = L_{mn} + \sum_{j=1}^{[m/2]} C_{m,j}(-1)^{nj+j-1} L_n^{m-2j} ,$$

where

$$C_{k,0} = 1,$$

$$C_{m,j} = C_{m-1,j} + C_{m-2,j-1}, \qquad 1 \le j \le [m/2] \text{ for } m \ge 2 .$$

Proof. The proof shall proceed by induction. For all n, the theorem is true for $m = 1$, the sum being empty. Assume, for $n \ge 1$,

$$L_n^k = L_{nk} + \sum_{j=1}^{[k/2]} C_{k,j}(-1)^{nj+j-1} L_n^{k-2j}$$

for $k = 1, 2, 3, \cdots, m$ along with

$C_{k,0} = 1, \quad C_{2k,k} = 2, \quad C_{2k+1,k} = 2k+1, \quad$ and $\quad C_{2k+1,k+1} = 0.$

Therefore,

$$L_n^m = L_{mn} + \sum_{j=1}^{[m/2]} C_{m,j}(-1)^{nj+j-1}L_n^{m-2j} ,$$

and

$$L_n^{m+1} = L_nL_{mn} + \sum_{j=1}^{[m/2]} C_{m,j}(-1)^{nj+j-1}L_n^{m+1-2j} .$$

But,

$$L_nL_{mn} = L_{(m+1)n} + (-1)^nL_{(m-1)n} .$$

Thus,

$$L_n^{m+1} = L_{(m+1)n} + (-1)^nL_{(m-1)n} + \sum_{j=1}^{[m/2]} C_{m,j}(-1)^{nj+j-1}L_n^{m+1-2j}$$

Returning to the inductive assumption for $k = m - 1$ yields

$$\begin{aligned}(-1)^nL_{(m-1)n} &= (-1)^nL_n^{m-1} + (-1)^{n+1}\sum_{j=1}^{[(m-1)/2]} C_{m-1,j}(-1)^{nj+j-1}L_n^{m-1-2j} \\ &= (-1)^nL_n^{m-1} + \sum_{j=1}^{[(m-1)/2]} C_{m-1,j}(-1)^{n(j+1)+(j+1)-1}L_n^{m-1-2j} .\end{aligned}$$

Now let $p = j + 1$; then since $[(m-1)/2] + 1 = [(m+1)/2]$,

$$(-1)^n L_{(m-1)n} = (-1)^n L_n^{m-1} + \sum_{p=2}^{[(m+1)/2]} C_{m-1,p-1}(-1)^{np+p-1} L_n^{m+1-2p} .$$

Therefore,

$$L_n^{m+1} = L_{(m+1)n} + \left\{ (-1)^n L_n^{m-1} + \sum_{p=2}^{[(m+1)/2]} C_{m-1,p-1}(-1)^{np+p-1} L_n^{m+1-2p} \right\}$$

$$+ \sum_{p=1}^{[m/2]} C_{m,p}(-1)^{np+p-1} L_n^{m+1-2p}$$

$$= L_{(m+1)n} + \sum_{p=1}^{[(m+1)/2]} (C_{m,p} + C_{m-1,p-1})(-1)^{np+p-1} L_n^{m+1-2p} .$$

We examine the possible extra term added to the second summation. If m is $2k$, then $[m/2] = [(m + 1)/2] = k$ and $C_{2k,k} = 2$ and $C_{2k-1,k-1} = 2k - 1$; thus, $C_{2k+1,k} = 2k + 1$. If $m = 2k + 1$, then $[m/2] + 1 = [(m + 1)/2] = k + 1$ and the term $C_{2k+1,k+1} = 0$ and $C_{2k,k} = 2$; thus $C_{2k+2,k+1} = 2$. Thus, if one defines

$$C_{k-1,0} = 1, \quad C_{2k,k} = 2, \quad C_{2k+1,k} = 2k + 1, \quad C_{2k+1,k+1} = 0$$

for $k \ge 1$, and

$$C_{m+1,p} = C_{m,p} + C_{m-1,p-1}, \qquad 1 \le p \le \left[\frac{m+1}{2}\right], \qquad m \ge 1 ,$$

then

$$L_n^{m+1} = L_{(m+1)n} + \sum_{p=1}^{[(m+1)/2]} C_{m+1,p}(-1)^{np+p-1} L_n^{m+1-2p} ,$$

[Continued on p. 427.]