A NOTE ON FIBONACCI FUNCTIONS

W. R. SPICKERMAN
East Carolina University, Greenville, North Carolina

Recently, a number of authors [1, 2, 3] have considered Fibonacci functions — continuous functions possessing properties related to Fibonacci sequences. In this note, some Fibonacci functions are derived and their properties verified. The derivation is based on the following definition.

Definition: If f is an infinitely differentiable function and f satisfies the recursion relation:

(1) $$f(x + 2) = f(x) + f(x + 1),$$

then f is a Fibonacci function.

An immediate consequence of the definition is:

Theorem 1. If f(x) is a Fibonacci function, then $f'(x)$ and $\int f(x)dx$ are also.

The theorem is established by elementary calculus.

$$\begin{aligned} f'(x + 2) &= [f(x) + f(x + 1)]' = \\ &= f'(x) + f'(x + 1) \\ \int f(x + 2)dx &= \int [f(x) + f(x + 1)]dx = \\ &= \int f(x)\,dx + \int f(x + 1)dx. \end{aligned}$$

Theorem 2. If f(x) and g(x) are Fibonacci functions, then their sum is also.

Proof. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned} F(x + 2) = f(x + 2) + g(x + 2) &= [f(x + 1) + g(x + 1)] + [f(x) + g(x)] = \\ &= F(x + 1) + F(x)\,. \end{aligned}$$

Theorem 3. If f(x) is a Fibonacci function and c is a real constant, then cf(x) is a Fibonacci function.

Proof. Let $F(x) = cf(x)$. Then

$$F(x+2) = cf(x+2) = c[f(x+1) + f(x)] = cf(x+1) + cf(x) =$$
$$= F(x+1) + F(x) \;.$$

Since the function $e^{(p+k\pi i)x}$ where p is a real constant, k an integer, and $i = \sqrt{-1}$ is real for integer values of x, we look for Fibonacci functions of the form $y = e^{dx}$ where d is complex. Substitution into the recursion relation (1) yields

$$e^{d(x+2)} - e^{d(x+1)} - e^{dx} = 0 \tag{3}$$

or,

$$e^{dx}(d^{2d} - e^{d} - 1) = 0 \;. \tag{4}$$

Since 0 is omitted by the first factor of (4),

$$e^{2d} - e^{d} - 1 = 0 \;. \tag{5}$$

Solving (5) for e^d:

$$e^{d_1} = \tfrac{1}{2}(1 + \sqrt{5}) = \alpha \;,$$

and

$$e^{d_2} = \tfrac{1}{2}(1 - \sqrt{5}) = \beta \;.$$

Let $d_1 = a_1 + b_1 i$, then

$$a^{a_1}(\cos b_1 + i \sin b_1) = \alpha \;.$$

Since $\alpha > 0$, $a_1 = \ln\alpha = 0.48$ and $b_1 = 2k\pi$ for k an integer. Similarly, if $d_2 = a_2 + b_2 i$, then

$$e^{a_2}(\cos b_2 + i \sin b_2) = \beta \;.$$

Since $\beta < 0$, $a_2 = \ln|\beta|$ and $b_2 = (2k + i\ )$ for k an integer. Furthermore,

$$1 = \left|(e^{a_1} \cos 2k)(e^{a_2} \cos (2n + 1)\right| = \left|e^{a_1}\right|\left|e^{a_2}\right|$$

and so $a_2 = -\ln\alpha = -0.48$, or $a_2 = -a_1$. Thus, the subscript on a is not necessary and two solutions of (1) are:

$$y(x) = e^{ax} \cos 2k\pi x$$

and

$$y(x) = e^{-ax} \cos (2k + 1)\pi x$$

Applying Theorems 2 and 3, we have:

(6) $$y(x) = c_1 e^{ax} \cos 2k\pi x + c_2 e^{-ax} \cos (2n + 1)\pi x,$$

where $a = \ln\alpha$; k and n integers. Equation (6) may be written:

(7) $$y(x) = c_1 e^{(a+2k\pi i)x} + c_2 e^{(-a+(2n+1)\pi)x} .$$

Some interesting and useful relations between e^a and e^{-a} can be derived by substituting the values of d_1 and d_2 into Eq. (5).

$$e^{(a+2k\pi i)2} - e^{a+2k\pi i} - 1 = 0$$

$$e^{2a} e^{4k\pi i} - e^a e^{2k\pi i} - 1 = 0$$

$$e^{2a} - e^a - 1 = 0 ,$$

or

(8) $$e^{2a} = 1 + e^a$$

Also,

$$e^{(-a+(2k+1)\pi i)2} - e^{(-a+(2k+1)\pi i)} - 1 = 0$$

$$e^{-2a}e^{2(2k+1)\pi i} - e^{-a}e^{2k\pi i}e^{\pi i} - 1 = 0$$

$$e^{-2a} + e^{-a} - 1 - 0 \; ,$$

or

(9) $$e^{-2a} = 1 - e^{-a} \; .$$

Furthermore,

(10) $$e^{a} + e^{-a} = |\alpha| + |\beta| = \sqrt{5} \; .$$

The trigonometric identity $\cos k\pi(x+2) = \cos k\pi x = -\cos k\pi(x+1)$, relations (8) and (9), and some algebra verify that (6) is a solution to (1).

Since (6) is a differentiable function satisfying relation (1) in view of Theorem 1,

(11) $$y'(x) = (c_1e^{ax} \cos 2k\pi x + c_2e^{-ax} \cos (2n+1)\pi x)'$$

and

(12) $$\int y(x)dx = \int [c_1e^{ax} \cos 2k\pi x + c_2e^{-ax} \cos (2n+1)\pi x]dx$$

are also Fibonacci functions.

The values of c_1 and c_2 for which Eq. (6) assumes the Fibonacci numbers for integer x can be computed by applying the conditions $y(0) = 0$ and $y(1) = 1$. That is,

(13) $$\begin{aligned} c_1 + c_2 &= 0 \\ c_1e^{a} - c_2e^{-a} &= 1 \; . \end{aligned}$$

The solutions to the system (13) are $c_1 = 1/\sqrt{5}$ and $c_2 = -1\sqrt{5}$. Thus, the Fibonacci functions that agree with the Fibonacci numbers for integer x are

$$y(x) = (e^{ax} \cos 2k\pi x - e^{-ax} \cos (2n + 1)\pi x)/\sqrt{5} \ . \tag{14}$$

The function $f(x) = (a^x - b^x \cos \pi x)/\sqrt{5}$ [2] is a special case of (14), where $k = 0$, $n = 0$, and a^x and b^x are not identified as exponentials base e.

The usual extension of the Fibonacci sequence to the negative integers satisfies the relation $F_{-n} = (-1)^{n+1}F_n$. For integer values of x, the Fibonacci functions (14) have the same property.

Since

$$\cos 2kn\pi = (-1)^{2kn} = 1 \ ,$$

and

$$\cos (2kn + n)\pi = (-1)^{2kn}(-1)^n = (-1)^n \ ,$$

we have

$$\begin{aligned} \sqrt{5}\, y(-n) &= e^{-an} \cos 2k\pi(-n) - e^{an} \cos (2n + 1)\pi(-n) = \\ & e^{-an} - (-1)^n e^{an} = (-1)^{n+1}(e^{an} - e^{-an}) = \\ & = (-1)^{n+1} y(n)\sqrt{5} \ . \end{aligned}$$

REFERENCES

1. M. Elmore, "Fibonacci Functions," Fibonacci Quarterly, 5 (1967), pp. 371-382.
2. F. Parker, "A Fibonacci Function," Fibonacci Quarterly, 6 (1968), pp. 1-2.
3. A. Scott, "Continuous Extensions of Fibonacci Identities," Fibonacci Quarterly, 6 (1968), pp. 245-250.