INFINITELY MANY GENERALIZATIONS OF ABEL'S PARTIAL SUMMATION IDENTITY

KENNETH B. STOLARSKY*
Institute for Advanced Study, Princeton, New Jersey

It is well known that if $\Sigma_{k=1}^{m} B_k(x)$ is bounded independently of m and x (say for all x in an interval I) and A_k tends to zero monotonically as $k \to \infty$, then $\Sigma_{k=1}^{\infty} A_k B_k(x)$ is uniformly convergent on I. This follows from a finite identity first used systematically by Abel, namely,

$$(1)\qquad \sum_{k=m}^{n} A_k B_k = s_n A_n - s_{m-1} A_{m-1} + \sum_{k=m-1}^{n-1} s_k (A_k - A_{k+1}) ,$$

where

$$s_k = \sum_{i=1}^{k} B_i .$$

The purpose of this paper is to show that an infinite sequence of finite identities involving summations (of which (1) is the simplest example) can be deduced from the so-called "P. Hall commutator collecting process" which is fundamental in the theory of finitely generated nilpotent groups.

Let G be the free group on two generators a and b, $\{G_n\}$ its lower central series ($G_1 = G$, $G_{n+1} = [G_n, G]$), and $\{\phi_n\}$ the corresponding natural homomorphisms, so $\phi_n : G \to G/G_n$. P. Hall's commutator collecting process yields for every $g \in G$ an integer $r = r(n)$ such that

$$(2)\qquad \phi_n(g) = c_1^{e_1} c_2^{e_2} \cdots c_r^{e_r} G_n ,$$

*Supported in part by a WARF fellowship, and also by National Science Foundation Grant GP-7952X.

where the $c_i \in G$ are the so-called basic commutators ($c_1 = a$, $c_2 = b$, $c_3 = [b,a]$, $c_4 = [b,a,a]$, $c_5 = [b,a,b]$, $c_6 = [b,a,a,a]$, $c_7 = [b,a,a,b]$, $c_8 = [b,a,b,b], \cdots$) and the e_i are integers uniquely determined by g and n. A detailed explanation of these concepts can be found in Chapter 5 of Magnus, Karrass, and Solitar, Combinatorial Group Theory, Interscience Publishers, John Wiley and Sons, Inc., New York, 1966.

Now let * denote the operator on G which turns words backwards; e.g., $(a^3b^2ab)^* = bab^2a^3$. If $\phi_n(g)$ is given by (2), define $\phi_n^*(g)$ by

(3) $$\phi_n^*(g) = c_1^{*e_1} c_2^{*e_2} \cdots c_r^{*e_r} G_n .$$

Since g^* can be formed by making the substitutions $a \to a^{-1}$ and $b \to b^{-1}$ in g^{-1}, it follows that

(4) $$\phi_n^*(g) = \phi_n(g^*) .$$

Similarly, let ' denote the operator on G which interchanges a and b; e.g., $(a^3b^2ab)' = b^3a^2ba$. Then,

(5) $$\phi_n'(g) = \phi_n(g') .$$

Equations (4) and (5) provide infinitely many generalizations of (1).

To obtain specific identities from (4) and (5) write g in the form

(6) $$g = b^{x_0} a^{x_1} b^{x_2} \cdots b^{x_{2m}}$$

where the x_i are integers. Then g^* is obtained from (6) by replacing x_i with x_{2m-i}, and g' is similarly obtained by replacing m with m + 1 and x_i with y_i, where $y_0 = y_{2m+2} = 0$ and $y_i = x_{i-1}$ for $1 \le i \le 2m + 1$. Tables I, II, and III of the appendix show how to calculate $\phi_5(g)$ from g, where g has the form (6), and $\phi_5^*(g)$, $\phi_5'(g)$ from $\phi_5(g)$, where $\phi_5(g)$ has the form (2).

Example 1. By equating the exponents of c_3 in (5), we obtain

(7)
$$\left(\sum_{i=0}^{m-1} x_{2i+1}\right)\left(\sum_{i=0}^{m} x_{2i}\right) - \sum_{i=0}^{m-1}\left(\sum_{j=i}^{m-1} x_{2j+1}\right)x_{2i}$$
$$= \sum_{i=0}^{m+1} y_{2i}\left(\sum_{j=i}^{m+1} y_{2j+i}\right) = \sum_{i=1}^{m}\left(\sum_{j=i}^{m} x_{2j}\right)x_{2i-1} .$$

By letting

$$u(t) = \sum_{i=0}^{t} x_{2i}$$

and

$$v(t) = \sum_{i=0}^{t} x_{2i+1} ,$$

this may be expressed in the more familiar form

(8)
$$u(m - 1)v(m - 1) - \sum_{i=0}^{m-1} v(i - 1)(u(i) - u(i - 1)) =$$
$$= \sum_{i=1}^{m} u(i - 1)(v(i - 1) - v(i - 2)) ,$$

which is the discrete analogue of the familiar $uv - \int vdu = \int udv$. Equation (1) is easily verified from (8).

Example 2. By equating the exponents of c_5 in (4), we obtain

$$\sum_{i=1}^{m} x_{2i}\left(\sum_{j=0}^{i-1} x_{2j}\right)\left(\sum_{j=0}^{i-1} x_{2j+1}\right) + \sum_{i=0}^{m}\binom{x_{2i}}{2}\left(\sum_{j=0}^{i-1} x_{2j+1}\right)$$

$$(9) \qquad = \sum_{i=0}^{m-1} x_{2i}\left(\sum_{j=i+1}^{m} x_{2j}\right)\left(\sum_{j=i}^{m-1} x_{2j+1}\right) + \sum_{i=0}^{m}\binom{x_{2i}}{2}\left(\sum_{j=i}^{m-1} x_{2j+1}\right)$$

$$+\left[\frac{1}{2}\left(\sum_{j=0}^{m-1} x_{2j+1}\right)\left(\sum_{j=0}^{m} x_{2j}\right) - \sum_{i=0}^{m} x_{2i}\left(\sum_{j=i}^{m-1} x_{2j+i}\right)\right]\left(\sum_{j=0}^{m} x_{2j} - 1\right).$$

It follows by equating the coefficients of x_1 in (9) that

$$(10) \qquad \sum_{i=1}^{m} x_{2i}\left(\sum_{j=0}^{i-1} x_{2j}\right) + \sum_{i=0}^{m}\binom{x_{2i}}{2} = \frac{1}{2}\left(\sum_{j=0}^{m} x_{2j}\right)\left(\sum_{j=0}^{m} x_{2j} - 1\right).$$

Equation (10) is also directly obvious, and can be considered a discrete analogue of

$$\int u\,du = \frac{1}{2}u^2 .$$

It is clear that these identities provide new tests for the convergence of infinite series, but the author has neither been able to use them to decide the convergence of any series whose convergence is presently unknown, nor to show that these identities always have integral analogues.

APPENDIX

For a g given in the form (6), Table 1 gives the exponents e_i of (2) for $r(5) = 8$, and Tables 2 and 3 give the exponents f_i and h_i of $\phi_5^*(g)$ and $\phi_5^!(g)$, respectively. If p is a complicated expression, $(p)_q$ shall denote the binomial coefficient $\binom{p}{q}$. The author has extended these tables (by hand) to $r(6) = 14$. The formula for e_{14} is an unwieldy sum of five terms, one of which is

$$\sum_{i=0}^{m-1} (p)_2 \left(\sum_{j=i+1}^{m} x_{2j} \right) ,$$

where

$$p = x_{2i} \sum_{j=i}^{m-1} x_{2j+1} .$$

TABLE 1

$$e_1 = \sum_{i=0}^{m-1} x_{2i+1} \qquad e_2 = \sum_{i=0}^{m} x_{2i}$$

$$e_3 = \sum_{i=0}^{m-1} x_{2i} \left(\sum_{j=i}^{m-1} x_{2j+1} \right) \qquad e_4 = \sum_{i=0}^{m=1} x_{2i} \left(\sum_{j=i}^{m-1} x_{2j+1} \right)_2$$

$$e_5 = \sum_{i=0}^{m-1} x_{2i} \left(\sum_{j=i}^{m-1} x_{2j+1} \right) \left(\sum_{j=i+1}^{m} x_{2j} \right) + \sum_{i=0}^{m-1} \binom{x_{2i}}{2} \left(\sum_{j=i}^{m-1} x_{2j+1} \right)$$

$$e_6 = \sum_{i=0}^{m-1} x_{2i} \left(\sum_{j=i}^{m-1} x_{2j+1} \right)_3$$

$$e_7 = \sum_{i=0}^{m-1} \binom{x_{2i}}{2} \left(\sum_{j=i}^{m-1} x_{2j+1} \right)_2 + \sum_{i=0}^{m-1} x_{2i} \left(\sum_{j=i}^{m-1} x_{2j+1} \right)_2 \left(\sum_{j=i+1} x_{2j} \right)$$

$$e_8 = \sum_{i=0}^{m-1} \binom{x_{2i}}{3} \left(\sum_{j=i}^{m-1} x_{2j+1} \right) + \sum_{i=0}^{m-1} \binom{x_{2i}}{2} \left(\sum_{j=i}^{m-1} x_{2j+1} \right) \left(\sum_{j=i+1}^{m} x_{2j} \right)$$

$$+ \sum_{i=0}^{m-1} x_{2i} \left(\sum_{j=i}^{m-1} x_{2j+i} \right) \left(\sum_{j=i+1}^{m} x_{2j} \right)_2 .$$

[Continued on p. 405.]