SEQUENCES WITH A CHARACTERISTIC NUMBER

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1. A Fibonacci sequence $a_0, a_1, a_2, \cdots, a_n, \cdots$ is called a Fibonacci sequence if it satisfies the recursion relation

$$a_{n+2} = a_{n+1} + a_n \quad . \tag{1}$$

A well-known property of such a sequence is that there exists a number α such that

$$a_n a_{n+2} - a_{n+1}^2 = (-1)^n \alpha \tag{2}$$

for all $n = 0, 1, 2, \cdots$. The number α is called the characteristic number of the sequence [1]. The purpose of this paper is to explore the significance of the characteristic number [2] and to identify all sequences that have a characteristic number. We shall consider only sequences of rational numbers.

2. We call a sequence geometric if there exist numbers a and r such that

$$a_n = ar^n, \qquad n = 0, 1, 2, \cdots \quad . \tag{3}$$

If a sequence is geometric, then

$$a_n a_{n+2} - a_{n+1}^2 = 0, \qquad n = 0, 1, 2, \cdots \quad . \tag{4}$$

Conversely, suppose Eq. (4) holds. If $a_n \neq 0$ for all n, then

$$\frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1}}{a_n}, \qquad n = 0, 1, 2, \cdots \quad . \tag{5}$$

Then the sequence satisfies (3) with $a = a_0$, and

$$r = \frac{a_1}{a_0} .$$

If $a_n = 0$ for some n, then by Eq. (4), $a_{n+1} = 0$. If $n \geq 2$, then by Eq. (4),

$$a_{n-2}a_n - a_{n-1}^2 = 0,$$

and $a_{n-1} = 0$. Hence, if $a_n = 0$ for some n, then $a_n = 0$ for all $n = 1, 2, 3, \cdots$. That is, either every term of the sequence is 0, or only a_0 is not 0. In the first case, the sequence satisfies Eq. (3) with $a = 0$, and r arbitrary. In the second case, it satisfies Eq. (3) with $a = a_0$, and $r = 0$. Therefore, a sequence is geometric if and only if it satisfies Eq. (4). Equation (4) is a special case of Eq. (2) with $d = 0$. Since Eq. (2), with $d \neq 0$ represents a minor deviation from the typical behavior of a geometric sequence, we shall call any sequence satisfying Eq. (2) with $d \neq 0$ a <u>parageometric sequence.</u>

3. We shall call a sequence <u>almost geometric</u> if it is not geometric, but there exist numbers r_n such that $a_{n+1} = a_n r_n$ for $n = 0, 1, 2, \cdots$, and the sequence (r_n) approaches a limit as n becomes infinite. For example, in the Fibonacci sequence defined by

$$(6) \qquad F_0 = 1, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n = 0, 1, 2, \cdots,$$

the terms of the sequence are given by the Binet formula

$$(7) \qquad F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}$$

Then

$$r_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} = \frac{\alpha - \beta\left(\frac{\beta}{\alpha}\right)^n}{1 - \left(\frac{\beta}{\alpha}\right)^n}$$

$$= \frac{\alpha + \beta\left(\frac{1}{\alpha^2}\right)^n (-1)^{n+1}}{1 + \left(\frac{1}{\alpha^2}\right)^n (-1)^{n+1}} .$$

But

$$\frac{1}{\alpha^2} = \frac{4}{6 + 2\sqrt{5}} < 1 .$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{\alpha^2}\right)^n = 0 ,$$

and $\lim_{n\to\infty} r_n = \alpha$. So the Fibonacci sequence, defined by (6), which is parageometric with $d = 1$, is also almost geometric.

4. We shall call a sequence <u>alternating</u> if $a_{2k} = a$, $a_{2k+1} = b$, $a \neq b$, for all $n = 0, 1, 2, \cdots$. An alternating sequence satisfies Eq. (2) with $d = a^2 - b^2$. Then $d = 0$ if and only if $b = -a$. So, an alternating sequence is geometric if and only if $b = -a$, and it is parageometric in all other cases. However, <u>a parageometric alternating sequence is not almost geometric.</u> In fact, if $a = 0$ and $b \neq 0$, then r_n cannot be defined for even n. If $a \neq 0$ and $b = 0$, then r_n cannot be defined for odd n. If neither a nor b is zero, then

$$r_n = \frac{b}{a}$$

for even n, and

$$r_n = \frac{a}{b}$$

for odd n, and

$$\frac{a}{b} \neq \frac{b}{a} \quad ,$$

so, while r_n is defined for all n, it does not approach a limit as n becomes infinite. Hence, every alternating sequence is not almost geometric.

5. We shall call a sequence eventually almost geometric if it is not almost geometric, but the sequence obtained by deleting the first k terms, for some positive integer k, is almost geometric. For example, the sequence $0, 1, 0, 1, 0, a_5, a_6, a_7, \cdots$, where $a_{m+5} = F_m$ for $m = 0, 1, 2, \cdots$, is parageometric and is not almost geometric, but it is eventually almost geometric. Similarly, the sequence $8, 5, 3, 2, 1, 1, 0, 1, 0, 1, 0, a_{11}, a_{12}, \cdots$, where $a_{m+11} = F_m$ for $m = 0, 1, 2, \cdots$, is parageometric, is not almost geometric, but it is eventually almost geometric.

We shall call a sequence eventually alternating if it is not alternating, but the sequence obtained by deleting the first k terms, for some positive integer k, is alternating. For example, the sequence $8, 5, 3, 2, 1, 1, a_7, a_8, \cdots$, where a_{6+n} is 0 for odd n, and is 1 for even n, is parageometric, is not alternating, but is eventually alternating.

6. We can now state our principal result.

Theorem. If a sequence is not geometric, and no term of the sequence is 0, it is parageometric if and only if it satisfies the recursion relation

$$a_{n+2} = k a_{n+1} + a_n \tag{8}$$

for some rational number k. If $k = 0$, the sequence is alternating, and if $k \neq 0$, the sequence is almost geometric.

A zero term may occur in the sequence only if the absolute value of its characteristic number is a perfect square. If there is a zero term in the sequence, then either the sequence is alternating, or the sequence is eventually alternating, or the sequence is eventually almost geometric. In the first case, the sequence satisfies the recursion relation (8) with $k = 0$. In the second case, for some index $i > 0$, $a_0, a_1, \cdots, a_i$ is a fragment of an almost geometric sequence satisfying the recursion relation (8) for some $k \neq 0$, and

$a_{i+1}, a_{i+2}, \cdots, a_{i+n}, \cdots$ is an alternating sequence satisfying (8) with $k = 0$. In the third case, there are two possibilities: (1) For some index $j > 0$, $a_0, a_1, \cdots, a_j$ is a fragment of an alternating sequence satisfying the recursion relation (8) with $k = 0$, and $a_{i+1}, a_{i+2}, \cdots, a_{j+n}, \cdots$ is an almost geometric sequence satisfying (8) for some $k \neq 0$. (2) For some non-negative index i, $a_0, a_1, \cdots, a_i$ is a fragment of an almost geometric sequence satisfying the recursion relation (8) for some $k \neq 0$; for some positive index $j > i$, $a_i, a_{i+1}, \cdots, a_j$ is a fragment of an alternating sequence satisfying the recursion relation (8) with $k = 0$; and $a_{j+1}, a_{j+2}, \cdots, a_{j+n}, \cdots$ is an almost geometric sequence satisfying (8) for some $k \neq 0$. Consequently, a parageometric sequence consists of at most three consecutive segments each of which satisfies the recursion relation (8) for some value of k.

Proof. (1) Let (a_n) be a sequence that is not geometric and with $a_n \neq 0$ for all $n = 0, 1, 2, \cdots$. If it is parageometric, we have

$$a_n a_{n+2} - a_{n+1}^2 = (-1)^n d .$$

Then

$$a_{n+1} a_{n+3} - a_{n+2}^2 = (-1)^{n+1} d .$$

Therefore,

$$a_n a_{n+2} - a_{n+1}^2 + a_{n+1} a_{n+3} - a_{n+2}^2 = 0 .$$

Hence

$$a_{n+1}(a_{n+3} - a_{n+1}) = a_{n+2}(a_{n+2} - a_n) .$$

Then, since $a_n \neq 0$ for all n,

$$\frac{a_{n+3} - a_{n+1}}{a_{n+2}} = \frac{a_{n+2} - a_n}{a_{n+1}} .$$

Thus

$$\frac{a_{n+2} - a_n}{a_{n+1}} = k$$

for some rational constant k and all values of $n = 0, 1, 2, \cdots$. Then (a_n) satisfies Eq. (8).

Conversely, suppose the sequence satisfies (8). Then

$$\frac{a_{n+3} - a_{n+1}}{a_{n+2}} = \frac{a_{n+2} - a_n}{a_{n+1}} = k \;.$$

Consequently,

$$a_{n+1}a_{n+3} - a_{n+2}^2 = -(a_n a_{n+2} - a_{n+1}^2)$$

for all $n = 0, 1, 2, \cdots$. If we let $d = a_0a_2 - a_1^2$, then we have

$$a_n a_{n+2} - a_{n+1}^2 = (-1)^n d \,.$$

Since the sequence is not geometric, $d \neq 0$, and the sequence is parageometric. If $k = 0$, then $a_{n+2} = a_n$. Since the sequence is not geometric, $a_n \neq a_{n+1}$. Hence it is alternating. The characteristic equation associated with (8) is

$$x^2 - kx - 1 = 0 \,, \tag{9}$$

whose roots are

$$r = \frac{k + \sqrt{k^2 + 4}}{2} \quad , \quad s = \frac{k - \sqrt{k^2 + 4}}{2} \quad . \tag{10}$$

Then, by the theory of linear recurrence relations [3],

$$a_n = ar^n + bs^n, \tag{11}$$

where a and b have the values

$$a = \frac{a_1 - a_0 s}{\sqrt{k^2 + 4}}, \quad b = \frac{a_0 r - a_1}{\sqrt{k^2 + 4}}. \tag{12}$$

$$d = a_0 a_2 - a_1^2 = (a + b)(ar^2 + bs^2) - (ar + bs)^2 = ab(r - s)^2 = ab(k^2 + 4).$$

Since $d \neq 0$, it follows that $a \neq 0$ and $b \neq 0$. If $k > 0$,

$$\left|\frac{s}{r}\right| < 1.$$

If $k < 0$,

$$\left|\frac{r}{s}\right| < 1.$$

$$r_n = \frac{a_{n+1}}{a_n} = \frac{ar^{n+1} + bs^{n+1}}{ar^n + bs^n} = \frac{r + \frac{b}{a} s\left(\frac{s}{r}\right)^n}{1 + \frac{b}{a}\left(\frac{s}{r}\right)^n} = \frac{\frac{a}{b} r\left(\frac{r}{s}\right)^n + s}{\frac{a}{b}\left(\frac{r}{s}\right)^n + 1}.$$

If $k > 0$,

$$\left|\frac{s}{r}\right| < 1,$$

and $\lim_{n \to \infty} r_n = r$. If $k < 0$,

$$\left|\frac{r}{s}\right| < 1,$$

and $\lim_{n \to \infty} r_n = s$. Consequently, if $k \neq 0$, the sequence is almost geometric.

(2) If some term $a_k = 0$, then

$$a_k a_{k+2} = a_{k+1}^2 = (-1)^n d ,$$

and hence

$$-a_{k+1}^2 \cdot = (-1)^k d .$$

If k is odd, d is a perfect square. If k is even, $-d$ is a perfect square. Since $d \neq 0$, $a_{n+1} \neq 0$. If $k \neq 0$, we have

$$a_{k-1} a_{k+1} - a_k^2 = (-1)^{n-1} d ,$$

or

$$a_{k-1} a_{k+1} = (-1)^{k-1} d .$$

Then

$$a_{k-1} a_{k+1} = a_{k+1}^2 ,$$

and

$$a_{k+1} (a_{k-1} - a_{k+1}) = 0 .$$

Then, since $a_{k+1} \neq 0$, $a_{k-1} = a_{k+1}$. That is, every zero term in the sequence is flanked by a pair of equal non-zero terms. Consequently, if $a_n = a_m = 0$, with $k < m$, then $m - k > 1$. If $a_k = 0$, it is possible that $a_{k+2} = 0$, and $a_{n-2} = 0$ if it exists. Then a_k belongs to a sequence of alternate zero terms

$$a_{k-2\ell} = a_{k-2+2\ell} = \cdots = a_{k-2} = a_k = a_{k+2} = \cdots = a_{k+2m} = 0 ,$$

where $\ell \geq 0$, $2\ell \leq k$, and $m \geq 0$.

$$(a_{k-2\ell-1}) = a_{k-2\ell+1} = \cdots = a_{k-1} = a_{k+1} = \cdots = a_{k+2m+1} \neq 0 ,$$

where the parentheses around the term $a_{k-2\ell-1}$ indicate that it is included only if it exists. (That is, if $k - 2\ell \neq 0$.) Then $a_{k-2\ell-1}$, $a_{k-2\ell}$, $\cdots$, a_{k+2m+1}, which is a segment of the sequence (a_n), is an alternating sequence with zero terms alternating with non-zero terms. Let us extend this alternating sequence as far as we can to both lower and higher indices by including a_{k+2m+2} and a_{k+2m+3} if $a_{k+2m+2} = 0$, and by including $a_{k-2\ell-2}$ and $a_{k-2\ell-3}$ if they exist and $a_{k-2\ell-2} = 0$. Then the following four possibilities arise, depending on whether or not the alternating sequence begins with a_0 on the left and whether or not it terminates on the right:

I. The alternating sequence begins with a_0, and does not terminate.

II. The alternating sequence begins with a_i, $i > 0$, and does not terminate.

III. The alternating sequence begins with a_0, and terminates with a_j, $j > 0$.

IV. The alternating sequence begins with a_i, $i > 0$, and terminates with a_j, $j > i$.

In case I, the sequence (a_n) is an alternating sequence, with either the odd-numbered terms or the even-numbered terms equal to zero. That is, it has the form $0, a, 0, a, 0, a, \cdots$ or $a, 0, a, 0, a, 0, \cdots$, where $a \neq 0$. Such a sequence satisfies the recursion relation (8) with $k = 0$.

In case II, $a_i \neq 0$, $a_{i+1} = 0$, and $a_{i-1} \neq 0$. The infinite sequence $a_i, a_{i+1}, \cdots$ is an alternating sequence of the form $a, 0, a, 0, \cdots$. We shall show that for every $n < i$, $a_n \neq 0$.

In case III, $a_j \neq 0$, $a_{j-1} = 0$, and $a_{j+1} \neq 0$. The finite sequence a_0, $a_1, \cdots, a_j$ has the form $0, a, 0, a, \cdots, 0, a$ or $a, 0, a, 0, \cdots, 0, a$. We shall show that for every $n > j$, $a_n \neq 0$.

In case IV, $a_i \neq 0$, $a_{i+1} = 0$, $a_{i-1} \neq 0$, $a_j \neq 0$, $a_{j-1} = 0$, $a_{j+1} \neq 0$. The finite sequence $a_i, \cdots, a_j$ has the form $a, 0, a, 0, \cdots, 0, a$. We shall show that for every $n < i$ and every $n > j$, $a_n \neq 0$.

Suppose $a_j \neq 0$, $a_{j-1} = 0$, $a_{j+1} \neq 0$ (cases III and IV). We shall call these assumptions Assumptions A. We show that for every $n > j$, $a_n \neq 0$. From

$$a_{j-1}a_{j+1} - a_j^2 = (-1)^{j-1}d$$

we get $a_j^2 = (-1)^j d$.

(13) $$a_j a_{j+2} - a_{j+1}^2 = (-1)^j d = a_j^2 \; .$$

Therefore,

$$a_j(a_{j+2} - a_j) = a_{j+1}^2 \, .$$

We consider first the case where $a_j > 0$. Then, since $a_{j+1} \neq 0$, $a_{j+2} - a_j > 0$, and $a_{j+2} > a_j > 0$.

(14) $$a_{j+1}a_{j+3} - a_{j+2}^2 = (-1)^{j+1}d = -a_j^2 \; .$$

Therefore,

$$a_{j+1}a_{j+3} = a_{j+2}^2 - a_j^2 > 0 \; .$$

Then a_{j+3} is not zero, and has the same sign as a_{j+1}. From (13) and (14),

$$a_j a_{j+2} - a_{j+1}^2 + a_{j+1}a_{j+3} - a_{j+2}^2 = 0 \; .$$

Therefore,

$$a_{j+2}(a_{j+2} - a_j) = a_{j+1}(a_{j+3} - a_{j+1}) \, .$$

Hence $a_{j+3} - a_{j+1}$ has the same sign as a_{j+1} and a_{j+3}. This, if $a_{j+1} > 0$, $a_{j+3} > a_{j+1}$, and if $a_{j+1} < 0$, $a_{j+3} < a_{j+1}$. In either case, $|a_{j+3}| > |a_{j+1}| > 0$. Now we proceed by induction. Assume that $a_{j+2k} > a_{j+2k-2} > \cdots > a_j > 0$, that $a_{j+2n+1}, a_{j+2k+1}, a_{j+2k-3}, \cdots, a_{j+1}$ have the same sign, and that

$$|a_{j+2k+1}| > |a_{j+2k-1}| > \cdots > |a_{j+1}| > 0 \; .$$

(15) $$a_{j+2k}a_{j+2k+2} - a^2_{j+2k+1} = (-1)^{j+2k}d = (-1)^j d = a_j^2 .$$

(16) $$a_{j+2k-1}a_{j+2k+1} - a^2_{j+2k} = (-1)^{j+2k-1}d = -a_j^2 .$$

$$a_{j+2k}a_{j+2k+2} - a^2_{j+2k+1} + a_{j+2k-1}a_{j+2j+1} - a^2_{j+2k} = 0 .$$

$$a_{j+2k}(a_{j+2k+2} - a_{j+2k}) = a_{j+2k+1}(a_{j+2k+1} - a_{j+2k-1}) .$$

Then, since a_{j+2k+1}, a_{j+2k-1}, and $a_{j+2k+1} = a_{j+2k-1}$ have the same sign, and $a_{j+2k} > 0$, $a_{j+2k+2} - a_{j+2k} > 0$, and

$$a_{j+2k+2} > a_{j+2k} > \cdots > a_j > 0 .$$

(17) $$a_{j+2k+1}a_{j+2k+3} - a^2_{j+2k+2} = (-1)^{j+2k+1}d = (-1)^{j+1}d = -a_j^2 .$$

Therefore,

$$a_{j+2k+1}a_{j+2k+3} = a^2_{j+2k+2} - a_j^2 > 0 .$$

Therefore, a_{j+2k+1} and a_{j+2k+3} have the same sign. From (15) and (17), we get

$$a_{j+2k}a_{j+2k+2} - a^2_{j+2k+1} + a_{j+2k+1}a_{j+2k+3} - a^2_{j+2k+2} = 0.$$

Then

$$a_{j+2k+1}(a_{j+2k+3} - a_{j+2k+1}) = a_{j+2k+2}(a_{j+2k+2} - a_{j+2k}) .$$

Therefore, $a_{j+2n+3} - a_{j+2n+1}$ has the same sign as a_{j+2k+1}. Hence,

$$a_{j+2k+3},\ a_{j+2k+1},\ \cdots,\ a_{j+1}$$

have the same sign, and

$$\left|a_{j+2k+3}\right| > \left|a_{j+2k+1}\right| > \cdots > \left|a_{j+1}\right| > 0 .$$

If $a_j < 0$, a similar argument shows that

$$a_{j+2k} < a_{j+2k-2} < \cdots < a_j < 0, \quad a_{j+2k+1}, \quad a_{j+2k-1}, \quad \cdots, \quad a_{j+1}$$

have the same sign, and

$$\left|a_{j+2k+1}\right| > \left|a_{j+2k-1}\right| > \cdots > \left|a_{j+1}\right| > 0 .$$

Hence, for every $n > j$, $a_n \neq 0$.

Suppose $i > 0$, $a_i \neq 0$, $a_{i+1} = 0$, $a_{i-1} \neq 0$ (cases II and IV). We shall call these assumptions Assumptions B. Because of the symmetry with respect to i of the indices in the equation

$$a_{i-1}a_{i+1} - a_i^2 = (-1)^{i-1}d = (-1)^{i+1}d ,$$

and because Assumptions A are symmetrical to Assumptions B with respect to i if we write i instead of j in Assumptions A, the argument above proceeds just as well in the direction of decreasing indices. Hence, for every $n < i$, $a_n \neq 0$. Then by (1), in cases II and IV, the sequence $a_0, \cdots, a_{i-1}$ satisfies Eq. (8) for some $k \neq 0$, and is a finite segment of an almost geometric sequence; and in cases III and IV, the sequence $a_{j+1}, a_{j+2}, \cdots, a_{j+n}, \cdots$ satisfies Eq. (8) for some $k \neq 0$, and is an almost geometric sequence. This completes the proof of the theorem.

An example of case IV is given in Section 5. Another example is the sequence

$$58, \ 24, \ 10, \ 4, \ 2, \ 0, \ 2, \ 0, \ 2, \ 0, \ 2, \ 8, \ 34, \ 144, \ \cdots .$$

In this sequence, the characteristic number $\alpha = 4$. The sequence is made up of three consecutive segments:

I. 58, 24, 10, 4;

II. 2, 0, 2, 0, 2, 0, 2;

III. 8, 34, 144, $\cdots$;

where Segment I is a fragment of an almost geometric sequence satisfying the recurrence relation $a_{n+2} = -2a_{n+1} + a_n$. Segment II is a fragment of an alternating sequence satisfying the recurrence relation $a_{n+2} = a_n$; Segment III is an almost geometric sequence satisfying the recurrence relation $a_{n+2} = 4a_{n+1} + a_n$.

7. Consider the set of all almost geometric sequences satisfying the recurrence relation (8) with given $k \neq 0$. The associated characteristic equation is (9), where roots are r and s given in (10). If r and s are irrational, the theory of these sequences is analogous to that of rational Fibonacci sequences. For example, just as the set of all rational Fibonacci sequences can be given a field structure isomorphic to the field extension $R(\alpha)$ (see [4]), the set of all rational sequences satisfying the recurrence relation (8) with given $k \neq 0$ such that r is irrational can be given a field structure isomorphic to the field extension $R(r)$. In fact, we may represent each such sequence $a_0, a_1, \cdots$ by the ordered pair (a_0, a_1), since the sequence is fully determined by its first two terms and the recurrence relation (8). Then $(a_0, a_1) \mapsto a_0 + a_1 r$ is an isomorphism if we define addition and multiplication of sequences by

$$(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1) .$$

$$(a_0, a_1)(b_0, b_1) = (a_0 b_0 + a_1 b_1, a_0 b_1 + a_1 b_0 + k a_1 b_1) .$$

8. If

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 \ddots}}$$

is a continued fraction, the convergents

$$c_n = \frac{p_n}{q_n}$$

for $n = 1, 2, 3, \cdots$ are given by $p_0 = 1$, $q_0 = 0$, $p_1 = a_1$, $q_1 = 1$, and $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n > 1$ [5]. If we let $a_1 = a_2 = \cdots = k \neq 0$, where k is rational, then the equations take the form $p_0 = 1$, $q_0 = 0$, $p_1 = k$, $q_1 = 1$, and $p_n = kp_{n-1} + p_{n-2}$, $q_n = kq_{n-1} + q_{n-2}$ for $n > 1$. Moreover, $q_1 = p_0$, and $q_2 = kq_1 + a_0 = k = p_1$. Hence, for all $n > 0$, $q_n = p_{n-1}$. Then

$$c_n = \frac{p_n}{q_n} = \frac{p_n}{p_{n-1}}$$

for $n > 0$. In this case, $\lim_{n\to\infty} C_n = r$, where r is a root of $x^2 - kx - 1 = 0$. Moreover, the relation

$$p_{i+2}\, q_{i+1} - p_{i+1}\, q_{i+2} = (-1)^i$$

in this case takes the form

$$p_i\, p_{i+2} - p_{i+1}^2 = (-1)^i = (-1)^i d ,$$

where $d = 1$. Hence the sequence $p_1, p_2, \cdots, p_n, \cdots$ is a parageometric sequence with characteristic number 1, and is also an almost geometric sequence satisfying the recursion relation $p_{n+2} = kp_{n+1} + p_n$. If k is a positive integer, the sequence is related to the golden-type rectangle [6].

9. Every sequence that has a characteristic number d is either geometric (with $d = 0$) or parageometric (with $d \neq 0$). If it is parageometric, it consists of at most three consecutive segments, each of which satisfies the recursion relation (8) for some value of k. If it is a geometric sequence (ar^n), and $r \neq 0$, it satisfies the recursion relation (8) with $k = r = 1/r$. If $r = 0$, the sequence is $a, 0, 0, \cdots$, and is composed of two consecutive segments a and $0, 0, \cdots$, each of which trivially satisfies Eq. (8). Hence, every sequence that satisfies Eq. (2) and therefore has a characteristic number

α consists of at most three consecutive segments each of which satisfies Eq. (8) for some value of k.

Let us now consider any sequence satisfying Eq. (8), to see if it also satisfies Eq. (2) and hence has a characteristic number. If the sequence is geometric, it satisfies Eq. (2) with $d = 0$. If the sequence is not geometric, and no term of the sequence is 0, we have already shown in Section 6 that it satisfies Eq. (2) with $d \neq 0$. Suppose now that the sequence is not geometric and contains a term $a_j = 0$. Then the method of proof used in Section 6 breaks down. However, this case can be covered by a general proof that does not require that all terms of the sequence be different from 0.

Let $a_0, a_1, \cdots, a_{n-1}, \cdots$ be a sequence satisfying Eq. (8) for some value of K. Let $d = a_0a_2 - a_1^2$. Then, for $n = 0$, the sequence satisfies Eq. (2). We now proceed by induction. Assume

$$a_n a_{n+2} - a_{n+1}^2 = (-1)^n d$$

for some fixed n.

$$\begin{aligned} a_{n+1}a_{n+3} - a_{n+2}^2 &= a_{n+1}(ka_{n+2} + a_{n+1}) - a_{n+2}^2 \\ &= ka_{n+1}a_{n+2} - a_{n+2}^2 + a_{n+1}^2 \\ &= a_{n+2}(ka_{n+1} - a_{n+2}) + a_{n+1}^2 \\ &= a_{n+2}(-a_n) + a_{n+1} = -(a_na_{n+2} - a_{n+1}^2) \\ &= (-1)^{n+1}d\,. \end{aligned}$$

Hence, every sequence satisfying Eq. (8) also satisfies Eq. (2), and therefore has a characteristic number.

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$$\begin{vmatrix} 3 & -3 & 1 & 0 & \cdots \\ -1 & 3 & -3 & 1 & \cdots \\ 0 & -1 & 3 & -3 & \cdots \\ 0 & 0 & -1 & 3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n = \frac{1}{2}(n+1)(n+2) .$$

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