COMBINATIONS, COMPOSITIONS AND OCCUPANCY PROBLEMS

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INTRODUCTION

Let $r \leq k$ be positive integers. By a composition of k into r parts (an r-composition of k) we mean an ordered sequence of r positive integers (called the parts of the composition) where sum is k, i.e.,

$$(1) \qquad a_1 + a_2 + \cdots + a_r = k .$$

The length of the part a_i in (1) is a_i, $i = 1, \cdots, r$. We call k integers

$$(2) \qquad x_1 < x_2 < \cdots < x_k$$

chosen from $\{1, 2, \cdots, n\}$ a k-combination (choice) from n. A part of (2) is a sequence of consecutive integers not contained in a longer sequence of consecutive integers. The length of such a part is the number of integers contained in it. For example, the 6-combination 2, 3, 4, 6, 8, 9 from $n \geq 9$ consists of 3 parts (2, 3, 4), (6), and (8, 9) of lengths 3, 1, and 2, respectively.

A great deal of literature exists on restricted compositions and may be found in most standard texts, for example [7]. However, there does not seem to be much literature on restricted combinations, in particular on the notion of parts with respect to combinations. The notion of parts has been used in [2] and [6] (in disguised form) as preliminaries to solve certain permutation problems. A treatment of restricted combinations in itself seems worthwhile for the following reasons. First, as noted in paragraph 4, the study of certain occupancy problems (like objects into unlike cells) is shown to follow immediately from the study of restricted combinations. Although, of course, many occupancy problem results are well known, many of the results obtained in paragraphs 1 and 2 by elementary combinatorial methods are believed new and might not otherwise be readily obtained. In particular, they are relevant to the development of tests of randomness in two-dimensional arrays. Also, restricted combinations are useful in dealing with certain restricted sequences of Bernoulli trials. In paragraph 1, the simple connection between restricted compositions and restricted combinations is given. Although the results contained herein are perhaps of a technical and specialized nature, the approach is completely elementary.

Throughout this note, we take

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{(n-k)!k!} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases} .$$

1. Consider the following six symbols, each denoting the number of compositions of k into r parts with further restrictions where indicated.

List 1

1. $C(k,r)$, no other restrictions.
2. $C(k,r;\, w)$, each part $\le w$.
3. $C_e(k,r)$, each part of even length.
4. $C_e(k,r;\, w)$, each part of even length and each part $\le w$ (even).
5. $C_0(k,r)$, each part of odd length.
6. $C_0(k,r;\, w)$, each part of odd length and each part $\le w$ (odd).

Expressions for the above numbers are well known and may be obtained by combinatorial arguments or by considering the appropriate enumerator generating function in each case, as described by Riordan [7, p. 124].

Corresponding to the 6 restricted combination symbols given in List 1, we have the following six restricted combination symbols, each denoting the number of k-combinations from n with exactly r parts and with further restrictions as indicated.

List 2

1. $g(n,k;\, r)$, no further restrictions.
2. $g(n,k;\, r,w)$, each part $\le w$.
3. $g_e(n,k;\, r)$, each part of even length.
4. $g_e(n,k;\, r,w)$, each part of even length and each part $\le w$ (even).
5. $g_0(n,k;\, r)$, each part of odd length.
6. $g_0(n,k;\, r,w)$, each part of odd length and each part $\le w$ (odd).

Denote by C^i the restricted composition symbol in the i^{th} row of List 1, and g^i the i^{th} restricted combination symbol of List 2. Then

$$g^i = \binom{n-k+1}{r} C^i, \qquad i = 1, \cdots, 6. \tag{3}$$

To establish (3), note that a k-combination from n can be represented by $n - k$ symbols 0 and k symbols 1 arranged along a straight line, the symbol 0 representing an integer not chosen and a symbol 1 **representing** an integer chosen. Now place $n - k$ symbols 0 along a straight line forming $n - k + 1$ cells including one before the first zero and one after the last. Choose r of these cells in

$$\binom{n-k+1}{r}$$

ways. Now distribute the k symbols 1 into these cells with none empty in C^i ways. The result follows.

In fact, corresponding to a specified r-composition of k with r parts we have

$$\binom{n-k+1}{r}$$

k-combinations of n consisting of r parts with the same specifications and clearly

$$g(n,k;\ b_1,b_2,\cdots,b_u) = \binom{n-k+1}{r} C(k;\ b_1,b_2,\cdots,b_u), \tag{4}$$

where $g(n,k;\ b_1, b_2, \cdots, b_u)$ denotes the number of k-combinations from n, $C(k;\ b_1, b_2, \cdots, b_u)$ denotes the number of compositions of k, each consisting of exactly b_i parts of length i, $i = 1, 2, \cdots, n$ with

$$\sum_{i=1}^{u} ib_i = r.$$

A succession of a k-combination (2) is a pair x_i, x_{i+1} with $x_{i+1} - x_i = 1$. It is easy to see that a k-combination from n contains exactly s successions if and only if it contains exactly k - s parts. Hence, instead of describing the restricted choices by their parts, we may use succession conditions. The numbers

$$g(n,k;\ r) = \binom{n-k+1}{r}\binom{k-1}{r-1}$$

and g(n,k; k - s) are used in [2] and [6]. The numbers

$$\sum_{r=1} g^i, \qquad i = 1, \cdots, 6$$

give the number of combinations with the same restrictions as on the combinations counted in g^i, $i = 1, \cdots, 6$, but with the number of parts not being specified. Of course, the numbers

$$\sum_{r=1} g^i$$

may also be found by considering the appropriate generating function.

Recurrence relations and expressions for g(n,k; r) and g(n,k; r,w) may be found in [2] and [3]. We consider now some special restricted combinations.

The number of k-combinations from n, all parts even and $\leq$ w, is, for k,w even,

$$g_e(n,k;\ w) = \sum_{r=1} g_e(n,k;\ r,w) = \sum_{r=1}\binom{n-k+1}{r}C_e(k,r;\ w)$$

(5)

$$= \sum_{i=0}(-1)^i\binom{n-k+t-i}{n-k}\binom{n-k+1}{i},\ t = \frac{k-iw}{2}\ .$$

The number $g_e(n,k;\ w)$ is also the coefficient of x^k in the expression $(1 + x^2 + x^4 + \cdots + x^w)^{n-k+1}$. Taking w sufficiently large in (5), the number of k-combinations from n with all parts even is, for k even

(6) $$g_e(n,k) = \binom{n - k/2}{k/2} \qquad \text{with} \quad g_e(n,0) = 1,$$

and

(7) $$g_e(n) = \sum_{r=0} \binom{n - r}{r}$$

is the number of choices from n with all parts even. $[g_e(n) = F_{n+1}]$

In the case of combinations with odd parts only, we have for $w \geq 3$ and w odd,

(8)
$$g_0(n,k;\ w) = \sum_{r=1} g_0(n,k;\ r,w)$$
$$= \sum_{r=1}\sum_{i=0} (-1)^i \binom{r}{i}\binom{t - 1}{r - 1}\binom{n - k + 1}{r},$$

where $r \equiv k \pmod 2$ and

$$t = \frac{k + r - i(w + 1)}{2} .$$

The enumerator generating function of $g_0(n,k;\ w)$ is

$$(1 + x + x^3 + \cdots + x^w)^{n-k+1} .$$

(9) $$g_0(n,k;\ k) = \binom{n - k + 1}{k}$$

is the number of k-combinations from n, no two consecutive, the lemma of Kaplansky [5]. Taking w sufficiently large in (8), the number of k-choices from n, all parts odd, is

$$(10)\qquad g_0(n,k) = \sum_{i=0} \binom{k-i-1}{i}\binom{n-k+1}{k-2i}, \quad g_0(n,9) = 1 ,$$

and the number of choices from n with all parts odd is

$$(11)\qquad g_0(n) = \sum_{k=0} g_0(n,k) .$$

2. We also obtain the following relations. For $n \geq w + 3$,

$$(12)\qquad \begin{aligned} g_e(n,k;\, w) &= g_0(n-1,k;\, w) + g_e(n-2,k-2;\, w) \\ &\qquad - g_e(n-w-3,\, k-w-2;\, w) . \end{aligned}$$

For, if a k-choice from n with all parts even and $\leq w$ (even):

(i) does not contain n, then it is a k-choice from $n - 1$ with all parts even and $\leq w$, and there are $g_e(n-1),\, k;\, w)$ of these:

(ii) does contain n, then it must contain $n - 1$. Deleting the $n - 1$ and n we have a (k-2)-choice from $n - 2$ (with all parts even and $\leq w$ (even)) which does not contain all of $n - w - 1,\ n - w, \cdots, n - 2$, and there are $g_e(n-2,\, k-2;\, w) - g_e(n-w-3,\, k-w-2;\, w)$ of these.

Of course,

$$(13)\qquad g_e(n,k;\, w) = \begin{cases} 0 & k = w + 2 = n \\ g_e(n,k) & k < w + 2 = n \\ g_e(n,k) & k \leq w + 1 , \end{cases}$$

and hence, from (12) and (13),

$$g_e(n,k) = g_e(n-1,k) + g_e(n-2, k-2) . \tag{14}$$

The latter is also easily obtained by observing that a k-combination of n with all parts even either does not contain n or does contain n and necessarily n - 1.

In the case $n = w + 3$ and $k = w + 2$, (12) becomes (using (7)),

$$g_e(w+3, w+2; w) = w/2 .$$

This is easily verified directly by observing that the $(w + 3)$-choices from $1, 2, \cdots, w + 3$ with all parts even and $\leq w$ (even) are obtained by removing from $1, 2, \cdots, w + 3$ one of the $w/2$ integers $3, 5, \cdots, w + 1$.

Let $g_e(n; w)$ denote the total number of combinations from n with all parts even and $\leq w$ (even). Then

$$g_e(n; w) = \sum_{k=0} g_e(n,k; w) . \tag{15}$$

Using (12) and summing over k we have

$$g_e(n; w) = \begin{cases} g_e(n-1; w) + g_e(n-2; w) - g_e(n-w-3; w), & n \geq w+3 \\ g_e(n-1; w) + g_e(n-2; w) - 1, & n = w+2 . \end{cases} \tag{16}$$

Putting $n \leq w + 1$ in (16) or summing over all k in (14) we obtain

$$g_e(n) = g_e(n-1) + g_e(n-2), \quad g_e(0) = g_e(1) = 1 . \tag{17}$$

The numbers $g_e(n)$ are Fibonacci numbers. The Fibonacci numbers arise in other cases of restricted combinations. For example, if f(n) denotes the number of combinations from n, no two consecutive, and T(n) denotes the number of combinations from n with odd elements in odd position and even elements in even position, then [7, p. 17. problem 15].

$$g_e(n) = f(n - 1) = T(n - 1), \quad n > 0 .$$

Also, by considering those combinations which do not contain n, those combinations which contain n, n - 1 but not n - 2, those containing n, n - 1, n - 2, n - 3 but not n - 4, $\cdots$, etc., we obtain for $n > k \geq w$, w even, the relation

$$(18) \qquad g_e(n,k; w) = \sum_{r=0}^{w/2} g_e(n - 2r - 1, k - 2r; w) .$$

In the case of odd parts, a relation comparable to (10) is not readily obtainable. However, for $n > k \geq w$, w odd, we have

$$(19) \qquad g_0(n, k; w) = g_0(n - 1, k; w) + \sum_{r=1}^{w+1/2} g_0(n - 2r, k - 2r + 1; w) .$$

The first term on the right side counts those choices not containing n, the second term those choices containing n but not n - 1, the third term those choices containing n, n - 1, n - 2 but not n - 3, $\cdots$, etc., the last term those choices containing n, n - 1, $\cdots$, n - w + 1 but not n - w.

Denote by $g_0(n; w)$ the number of combinations from n with all parts odd and $\leq w$ (odd). Then, for $n > w$ (odd), summing (19) over k yields

$$(20) \qquad g_0(n; w) = g_0(n - 1; w) + \sum_{r=1}^{\overline{w+1}/2} g_0(n - 2r; w) .$$

Taking w sufficiently large in (19), we have for $n > k$,

$$(21) \qquad g_0(n,k) = g_0(n - 1,k) + \sum_{r=1} g_0(n - 2r, k - 2r + 1)$$

and

$$g_0(k,k) = \begin{cases} 0 & \text{if } k = 0, 2, 4, 6, \cdots \\ 1 & \text{if } k = 1, 3, 5, 7, \cdots \end{cases} .$$

For example,

$$g_0(2,1) = g_0(1,1) + g_0(0,0) = 1 + 1 = 2 .$$

Using (21), it is easily seen that

$$(22) \qquad g_0(n) = \begin{cases} g_0(n-1) + g_0(n-2) + g_0(n-4) + g_0(n-6) + \cdots + g_0(1) + 1 & \text{for } n \text{ odd and } n \geq 3, \\ g_0(n-1) + \sum_{r=1}^{n/2} g_0(n-2r) & \text{for } n \text{ even,} \end{cases}$$

with $g_0(0) = 1$ and $g_0(1) = 2$.

3. In a k-combination from $\{1, \cdots, n\}$ if we consider 1 and n as adjacent, then we obtain "circular" k-combinations from n. For example, the set $\{1, 2, 6, 8, 9, 12\}$ is a circular 6-combination from 12 consisting the 3 parts $\{12, 1, 2\}$, $\{6\}$, $\{8, 9\}$ of length 3, 1, and 2, respectively. Corresponding to the 6 symbols of List 2, we obtain 6 circular k-combination symbols denoted by h^i, $i = 1, \cdots, 6$. Then

$$(23) \qquad h^i = \frac{n}{n-k}\binom{n-k}{r}C^i, \qquad 0 < k < n .$$

This is easily established by noting the proof for

$$h(n, k; r, w) = \frac{n}{n-k}\binom{n-k}{r}C(k, r; w)$$

in [3]. The special case of $r = k$ and $i = 1$ in (23) gives

$$h(n,k; k) = \frac{n}{n-k}\binom{n-k}{k} ,$$

the number of k-combinations, no two consecutive, of $\{1, \cdots, n\}$ arranged in a circle, the lemma of Kaplansky [5]. The numbers

$$H(n) = \sum_{k=0}^{[n/2]} h(n,k;\ k) \quad ,$$

with $h(n, 0;\ 0) = 1$, have the relation

$$H(n) = H(n-1) + H(n-2)$$

for $n \geq 4$ with $H(2) = 3$ and $H(3) = 4$. [$H(n) = L_n$, the Lucas numbers.]

The relation between g_i and h_i is, of course, given by

$$g_i = \frac{(n-k)(n-k+1)}{n(n-k+1-r)} h_i, \qquad i = 1,\cdots,6, \qquad n-k \geq r\ .$$

4. In examining the proof of (3), it is clear that each of the numbers g_i and

$$\sum_{r=1} g_i, \qquad i = 1, \cdots, 6\ ,$$

may be interpreted as the number of ways of putting loke objects into $n - k$ 1 unlike cells subject to corresponding conditions. Putting $n = m + k - 1$ we are then placing k like objects into m unlike cells with the corresponding conditions. For example, $g(m + k - 1,\ k;\ r)$ is the number of ways of doing this such that exactly r of the m cells are occupied while

$$B(m,\ k;\ w) = \sum_{r=1} g_e(m + k - 1,\ k;\ r,\ w)$$

is the number where any occupied cell contains an even number, not greater than w, of the like objects. Using (18),

$$B(m, k; w) = \sum_{r=0}^{w/2} B(m - 1, k - 2r; w) ,$$

w even. In particular,

$$g(m + k - 1, k; m) = \binom{k - 1}{m - 1}$$

and

$$\sum_{r=1} g(m + k - 1, k; r) = \sum_{r=1} \binom{m}{r}\binom{k - 1}{r - 1} = \binom{m + k - 1}{m}$$

are the well known occupancy formulae [Riordan, 7, p. 92 and p. 102, Problem 8], the first having none of the m cells empty and the second having no restriction on the distributions of the k objects. Also, the numbers

$$g(m + k - 1, k; r) \quad \text{and} \quad \sum_{r=1} g(m + k - 1, k; r, w)$$

are treated as occupancy problems by Riordan [7, pp. 102-104, Problems 9, 13, and 14].

The restricted combinations also have applications to certain ballot and random walk problems. For example, in an election between two candidates, the probability that a certain candidate leads after n votes but does not obtain more than $u > 0$ runs of votes nor a run of votes of length greater than w is equal to

$$\frac{\displaystyle\sum_{r=1}^{u} \sum_{k=[n+2/2]} g(n, k; r, w)}{2^n} .$$

Finally, by noting that for $w > 1$,

$$\sum \binom{a_2}{a_1}\binom{a_3}{a_1} \cdots \binom{a_w}{a_{w-1}} = C(k,\ a_w;\ w) \quad , \tag{24}$$

the sum taken over all solutions $(a_1, \cdots, a_{w-1})$,

$$a_i \geq 0, \qquad \text{of} \qquad a_1 + a_2 + \cdots + a_{w-1} = k - a_w \ ,$$

many of the expressions in [1] are simplified. In particular for $w = k$ (24) becomes

$$\binom{a_2}{a_1}\binom{a_3}{a_2} \cdots \binom{a_k}{a_{k-1}} = C(k,\ a_k) = \binom{k-1}{a_k - 1} \ . \tag{25}$$

Upon change of variables and some elementary manipulation, (25) becomes Theorem 16 of [4],

$$\binom{n-1}{r-1} = \sum_{\substack{b_1 + 2b_2 + 3b_3 + \cdots + nb_n = n \\ b_1 + b_2 + \cdots + b_n = r, \quad b_i \geq 0}} \frac{r!}{b_1!\, b_2! \cdots b_n!}$$

for all natural numbers n and r.

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