GENERAL IDENTITIES FOR RECURRENT SEQUENCES OF ORDER TWO

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1. INTRODUCTION

Let W_0, W_1, $a \neq 0$, and $b \neq 0$ be arbitrary real numbers, and define

$$(1.1) \qquad W_{n+2} = aW_{n+1} - bW_n, \quad a^2 - 4b \neq 0, \qquad (n = 0, 1, \cdots),$$

$$(1.2) \qquad U_n = (\alpha^n - \beta^n)/(\alpha - \beta) \qquad (n = 0, 1, \cdots),$$

$$(1.3) \qquad V_n = \alpha^n + \beta^n \qquad (n = 0, 1, \cdots),$$

$$(1.4) \qquad W_{-n} = (W_0 V_n - W_n)/b^n \qquad (n = 0, 1, \cdots),$$

where $\alpha \neq \beta$ are roots of $x^2 - ax + b = 0$. If $W_0 = 0$ and $W_1 = 1$, then $W_n \equiv U_n$, $n = 0, 1, \cdots$; and if $W_0 = 2$ and $W_1 = a$, then $W_n \equiv V_n$, $n = 0, 1, \cdots$. Our first result is

Theorem 1. Let W_n and W_n^* be solutions of (1.1). Let r, m, and n be integers (+, -, or 0). Then, for $k = 0, 1, \cdots$,

$$(1.5) \qquad \sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{r+m}^{k-i} W_r^i W_{n+rk+im}^* = b^{rk} U_m^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j W_{n+j}^* .$$

Special cases of (1.5) are given by

Corollary 1.

$$(1.6) \qquad \sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+m}^{k-i} U_r^i U_{n+rk+im} = b^{rk} U_m^k U_n ,$$

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+m}^{k-i} U_r^i W_{n+rk+im} = b^{rk} U_m^k W_n , \tag{1.7}$$

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+m}^{k-i} U_r^i V_{n+rk+im} = b^{rk} U_m^k V_n , \tag{1.8}$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+m}^{2k-i} V_r^i U_{n+2kr+im} = (a^2 - 4b)^k b^{2kr} U_m^{2k} U_n , \tag{1.9}$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+m}^{2k-i} V_r^i W_{n+2kr+im} = (a^2 - 4b)^k b^{2kr} U_m^{2k} W_n , \tag{1.10}$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+m}^{2k-i} V_r^i V_{n+2kr+im} = (a^2 - 4b)^k b^{2kr} U_m^{2k} V_n , \tag{1.11}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{r+m}^{2k+1-i} V_r^i V_{n+(2k+1)r+im} = -(a^2 - 4b)^{k+1} b^{(2k+1)r} U_m^{2k+1} U_n , \tag{1.12}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{r+m}^{2k+1-i} V_r^i U_{n+(2k+1)r+im} = -(a^2 - 4b)^k b^{(2k+1)r} U_m^{2k+1} V_n . \tag{1.13}$$

Our next result related to Theorem 1 is

Theorem 2. Let W_n be a solution of (1.1). Let r, m, and n be integers (+, -, or 0). Then, for $k = 0, 1, \cdots,$ we have

$$V_m^k W_{kr+n}$$

$$= \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} (W_1 U_{n-2jm} - bW_0 U_{n-2jm-1})$$

(1.14)

$$+ \sum_{j=0}^{[(k-1)/2]} \binom{k}{2j+1} (a^2 - 4b)^j b^{(2j+1)m} V_{r+m}^{k-2j-1} U_r^{2j+1} (W_1 V_{n-(2j+1)m} - bW_0 V_{n-(2j+1)m-1})$$

(1.15)

$$V_m^k U_{kr+n} = \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} U_{n-2jm}$$

$$+ \sum_{j=0}^{[(k-1)/2]} \binom{k}{2j+1} (a^2 - 4b)^j b^{(2j+1)m} V_{r+m}^{k-2j-1} U_r^{2j+1} V_{n-(2j+1)m} ,$$

(1.16)

$$V_m^k V_{kr+n} = \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} V_{n-2jm}$$

$$+ \sum_{j=0}^{[(k-1)/2]} \binom{k}{2j+1} (a^2 - 4b)^{j+1} b^{(2j+1)m} V_{r+m}^{k-2j-1} U_r^{2j+1} U_{n-(2j+1)m} .$$

2. PROOF OF THEOREM 1

Let $W_n^* = S_1\alpha^n + S_2\beta^n$ and $W_n = C_1\alpha^n + C_2\beta^n$, $n = 0, 1, \cdots,$ where S_i and C_i, $i = 1, 2$, are arbitrary constants. Since $W_0 = C_1 + C_2$ and $W_1 = C_1\alpha + C_2\beta$, we readily find that

$$(2.1) \qquad (\alpha - \beta)^k C_1^k = (W_1 - \beta W_0)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j \beta^j ,$$

$$(2.2) \qquad (\alpha - \beta)^k C_2^k = (\alpha W_0 - W_1)^k = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j \alpha^j .$$

Let L denote the left-hand side of (1.5). Then, using the binomial theorem and representation of W_n^*, we have

$$L = S_1\alpha^{n+rk}(W_{r+m} - \alpha^m W_r)^k + S_2\beta^{n+rk}(W_{r+m} - \beta^m W_r)^k \quad . \tag{2.3}$$

Since

$$W_{r+m} - \alpha^m W_r = (\beta^m - \alpha^m)\beta^r C_2$$

and

$$W_{r+m} - \beta^m W_r = (\alpha^m - \beta^m)\alpha^r C_1 \quad ,$$

we obtain, using $\alpha\beta = b$, (2.1), (2.2), and (1.2),

$$\begin{aligned} L &= S_1 b^{rk}\alpha^n(\beta^m - \alpha^m)^k C_2^k + S_2 b^{rk}\beta^n(\alpha^m - \beta^m)^k C_1^k \\ &= b^{rk}U_m^k\{S_1\alpha^n(-1)^k(\alpha - \beta)^k C_2^k + S_2\beta^n(\alpha - \beta)^k C_1^k\} \\ &= b^{rk}U_m^k\sum_{j=0}^{k}(-1)^j\binom{k}{j}W_1^{k-j}W_0^j(S_1\alpha^{n+j} + S_2\beta^{n+j}) = R \quad , \end{aligned} \tag{2.4}$$

where R denotes the right-hand side of (1.5).

If $W_r \equiv U_r$ and $W_n^* \equiv U_n$, then $W_0 = 0$ and (1.5) gives the special case (1.6), noting that all terms in the right-hand sum of (1.5) vanish except for $j = 0$.

Since

$$W_n = W_0 U_{n+1} + (W_1 - aW_0)U_n \quad , \tag{2.5}$$

we obtain (1.7) from (1.6); and (1.8) from (1.7) when $W_n \equiv V_n$.

If $W_n \equiv V_n$ (i.e., $C_1 = C_2 = 1$), then (2.4), with $k = p$, gives

$$L = b^{rp} U_m^p (\alpha - \beta)^p (S_1\alpha^n(-1)^p + S_2\beta^n) . \tag{2.6}$$

Noting that $(\alpha - \beta)^2 = a^2 - 4b$, then (2.6), for $W_n^* \equiv U_n$ (i.e., $S_1 = -S_2 = 1/(\alpha - \beta)$), gives (1.9) for $p = 2k$ and (1.13) for $p = 2k + 1$. Using (2.5), we get (1.10) from (1.9); and (1.11) from (1.10) when $W_n \equiv V_n$.

If $W_n^* \equiv V_n$ (i.e., $S_1 = S_2 = 1$), then (2.6) gives (1.12) for $p = 2k + 1$.

If $a = -b = 1$, then $U_n \equiv F_n$, and (1.6) gives the identity of Halton [1, p. 34] as a special case.

3. PROOF OF THEOREM 2

Our method is a generalization of a proof used in the unpublished Master's thesis of Vinson [2, pp. 14-16]. If we treat α^r and β^r as the unknowns in the system $(\alpha - \beta)U_r = \alpha^r - \beta^r$ and $V_{r+m} = \alpha^m\alpha^r + \beta^m\beta^r$, we obtain

$$V_m\alpha^r = V_{r+m} + (\alpha - \beta)\beta^m U_r \quad\text{and}\quad V_m\beta^r = V_{r+m} - (\alpha - \beta)\alpha^m U_r .$$

Since $W_{kr+n} = C_1\alpha^n(\alpha^r)^k + C_2\beta^n(\beta^r)^k$, we obtain

$$V_m^k W_{kr+n} = C_1\alpha^n(V_{r+m} + (\alpha - \beta)\beta^m U_r)^k + C_2\beta^n(V_{r+m} - (\alpha - \beta)\alpha^m U_r)^k$$

$$= \sum_{i=0}^{k} \binom{k}{i} (\alpha - \beta)^i V_{r+m}^{k-i} U_r^i (\alpha\beta)^{mi} (C_1\alpha^{n-mi} + (-1)^i C_2\beta^{n-mi}) . \tag{3.1}$$

Now

$$\begin{aligned} C_1\alpha^{n-mi} &+ (-1)^i C_2\beta^{n-mi} \\ &= [(W_1 - \beta W_0)\alpha^{n-mi} + (-1)^i(\alpha W_0 - W_1)\beta^{n-mi}] / (\alpha - \beta) \\ &= W_1 \frac{(\alpha^{n-mi} - (-1)^i\beta^{n-mi})}{\alpha - \beta} - b W_0 \frac{(\alpha^{n-mi-1} - (-1)^i\beta^{n-mi-1})}{\alpha - \beta} . \end{aligned}$$

Since $(\alpha - \beta)^2 = a^2 - 4b$, we obtain (1.14) from (3.1) for $i = 2j$ and $i = 2j + 1$.

If $W_n \equiv U_n$, then $W_0 = 0$, $W_1 = 1$, and thus (1.14) gives (1.15). If $W_n \equiv V_n$, then $W_0 = 2$, $W_1 = a$, and thus (1.14) gives (1.16), noting that $V_n = aU_n - 2bU_{n-1}$ and that

$$aV_n - 2bV_{n-1} = 2V_{n+1} - aV_n = (a^2 - 4b)U_n .$$

4. EXTENDED RESULTS

Our next class of results are of a higher level order than Theorem 1, since we now essentially replace W_n^* in (1.5) by its cross-product with itself.

Theorem 3. Let W_n and W_n^* be solutions of (1.1). Let r, m, p, and n be integers (+, -, or 0). Then, for $k = 0, 1, \cdots,$

$$\left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{r+2m}^{k-i} W_r^i W_{p+im}^* W_{n+im}^*\right)(a^2 - 4b)$$

$$(4.1) \quad = b^{rk} U_{2m}^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j (W_1^{2*} V_{p+n-rk+j} - 2bW_0^* W_1^* V_{p+n-rk+j-1} + b^2 W_0^{2*} V_{p+n-rk+j-2})$$

$$- (W_1^{2*} - aW_0^* W_1^* + bW_0^{2*}) b^n V_{p-n} U_m^k (W_1 V_{r+m} - bW_0 V_{r+m-1})^k .$$

Corollary 3. In special cases of (4.1), we have

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{r+2m}^{k-i} W_r^i V_{p+im} V_{n+im}$$

$$(4.2) \quad = b^{rk} U_{2m}^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j V_{p+n-rk+j}$$

$$+ b^n V_{p-n} U_m^k (W_1 V_{r+m} - bW_0 V_{r+m-1})^k ,$$

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+2m}^{k-i} U_r^i V_{p+im} V_{n+im} \tag{4.3}$$
$$= b^{rk} U_{2m}^k V_{p+n-rk} + b^n V_{p-n} U_m^k V_{r+m}^k ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+2m}^{2k-i} V_r^i V_{p+im} V_{n+im} \tag{4.4}$$
$$= (a^2 - 4b)^k b^{2rk} U_{2m}^{2k} V_{p+n-2rk} + (a^2 - 4b)^{2k} b^n V_{p-n} U_m^{2k} U_{r+m}^{2k} ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{r+2m}^{2k+1-i} V_r^i V_{p+im} V_{n+im} \tag{4.5}$$
$$= -(a^2 - 4b)^{k+1} b^{r(2k+1)} U_{2m}^{2k+1} U_{p+n-r(2k+1)}$$
$$+ (a^2 - 4b)^{2k+1} b^n V_{p-n} U_m^{2k+1} U_{r+m}^{2k+1} ,$$

$$(a^2 - 4b) \sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{r+2m}^{k-i} W_r^i U_{p+im} U_{n+im} \tag{4.6}$$
$$= b^{rk} U_{2m}^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j V_{p+n-rk+j}$$
$$- b^n V_{p-n} U_m^k (W_1 V_{r+m} - bW_0 V_{r+m-1})^k ,$$

$$(a^2 - 4b) \sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+2m}^{k-i} U_r^i U_{p+im} U_{n+im} \tag{4.7}$$
$$= b^{rk} U_{2m}^k V_{p+n-rk} - b^n V_{p-n} U_m^k V_{r+m}^k ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+2m}^{2k-i} V_r^i U_{p+im} U_{n+im} \tag{4.8}$$

$$= (a^2 - 4b)^{k-1} b^{2rk} U_{2m}^{2k} V_{p+n-2rk} - (a^2 - 4b)^{2k-1} b^n V_{p-n} (U_m U_{r+m})^{2k} \quad ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{r+2m}^{2k+1-i} V_r^i U_{p+im} U_{n+im}$$

$$= -(a^2 - 4b)^k b^{r(2k+1)} U_{2m}^{2k+1} U_{p+n-r(2k+1)} \tag{4.9}$$

$$- (a^2 - 4b)^{2k} b^n V_{p-n} U_m^{2k+1} U_{r+m}^{2k+1} \quad .$$

Closely associated with Theorem 3 is

Theorem 4. Let W_n be a solution of (1.1). Let r, m, p, and n be integers (+, -, or 0). Then, for $k = 0, 1, \cdots,$

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{r+2m}^{k-i} W_r^i U_{p+im} V_{n+im}$$

$$= b^{rk} U_{2m}^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j U_{p+n-rk+j} \tag{4.10}$$

$$+ b^n U_{p-n} U_m^k (W_1 V_{r+m} - bW_0 V_{r+m-1})^k \quad .$$

Corollary 4. As special cases of (4.10), we have

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} U_{r+2m}^{k-i} U_r^i U_{p+im} V_{n+im} \tag{4.11}$$

$$= b^{rk} U_{2m}^k U_{p+n-rk} + b^n U_{p-n} U_m^k V_{r+m}^k \quad ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{r+2m}^{2k-i} V_r^i U_{p+im} V_{n+im}$$

$$(4.12) \qquad = (a^2 - 4b)^k b^{2rk} U_{2m}^{2k} U_{p+n-2rk} + (a^2 - 4b)^{2k} b^n U_{p-n} U_m^{2k} U_{r+m}^{2k} ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{r+2m}^{2k+1-i} V_r^i U_{p+im} V_{n+im}$$

$$(4.13) \qquad = -(a^2 - 4b)^k b^{r(2k+1)} U_{2m}^{2k+1} V_{p+n-r(2k+1)}$$

$$+ (a^2 - 4b)^{2k+1} b^n U_{p-n} U_m^{2k+1} U_{r+m}^{2k+1} .$$

<u>Remarks</u>. Since $U_{2n} = U_n V_n$, we note that for $p = n$, (4.10), (4.11), (4.12) and (4.13) reduce to special cases, respectively, of (1.5), (1.6), (1.9), and (1.13).

5. PROOF OF THEOREM 3

We readily find that

$$(5.1) \quad W^*_{p+im} W^*_{n+im} = S_1^2 \alpha^{p+n} \alpha^{2mi} + S_2^2 \beta^{p+n} \beta^{2mi} + S_1 S_2 b^n V_{p-n} b^{mi} .$$

Let $(a^2 - 4b) \cdot L$ denote the left-hand side of (4.1). Then, the binomial theorem, using (5.1), gives

$$(5.2) \quad \begin{aligned} L = {} & S_1^2 \alpha^{p+n} (W_{r+2m} - \alpha^{2m} W_r)^k + S_2^2 \beta^{p+n} (W_{r+2m} - \beta^{2m} W_r)^k \\ & + S_1 S_2 b^n V_{p-n} (W_{r+2m} - b^m W_r)^k . \end{aligned}$$

Since $W_n = C_1 \alpha^n + C_2 \beta^n$, we have, using (2.1) and (2.2) for $k = 1$,

$$(5.3) \quad (W_{r+2m} - b^m W_r)^k = U_m^k (W_1 V_{r+m} - b W_0 V_{r+m-1})^k \equiv Y .$$

Noting the relations cited after (2.3), we have

$$
\begin{aligned}
L &= S_1^2 b^{rk} \alpha^{p+n-rk} C_2^k (\beta^{2m} - \alpha^{2m})^k \\
&\quad + S_2^2 b^{rk} \beta^{p+n-rk} C_1^k (\alpha^{2m} - \beta^{2m})^k + S_1 S_2 b^n V_{p-n} Y \qquad (5.4)\\
&= b^{rk} U_{2m}^k \left\{ S_1^2 \alpha^{p+n-rk} (-1)^k (\alpha - \beta)^k C_2^k + S_2^2 \beta^{p+n-rk} (\alpha - \beta)^k C_1^k \right\} \\
&\quad + S_1 S_2 b^n V_{p-n} Y \ .
\end{aligned}
$$

Recalling (2.1) and (2.2), we now have

$$
\begin{aligned}
L &= b^{rk} U_{2m}^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_0^j (S_1^2 \alpha^{p+n-rk+j} + S_2^2 \beta^{p+n-rk+j}) \qquad (5.5)\\
&\quad + S_1 S_2 b^n V_{p-n} Y \ .
\end{aligned}
$$

Since $(\alpha - \beta)S_1 = W_1^* - \beta W_0^*$ and $(\alpha - \beta)S_2 = \alpha W_0^* - W_1^*$, additional simplification of (5.5), using $\alpha\beta = b$, $\alpha + \beta = a$, and $(\alpha - \beta)^2 = a^2 - 4b$, yields (4.1).

If $W_n^* \equiv V_n$, then $W_0^* = 2$ and $W_1^* = a$, and thus (4.1) gives (4.2), noting that

$$a^2 V_c - 4ab V_{c-1} + 4b^2 V_{c-2} = (a^2 - 4b) V_c \ .$$

We get (4.3) from (4.2) when $W_n \equiv U_n$.

If $W_n \equiv V_n$, then (4.2) gives (4.4) and (4.5), which are also obtained from (5.4), where $S_1 = S_2 = C_1 = C_2 = 1$.

If $W_n^* \equiv U_n$, then (4.1) gives (4.6), which gives (4.7) for $W_n \equiv U_n$. If $W_n \equiv V_n$, then (4.6) gives (4.8) and (4.9), which are also obtained from (5.4), where now $C_1 = C_2 = 1$ and $S_1 = -S_2 = (\alpha - \beta)^{-1}$.

6. PROOF OF THEOREM 4

We readily find that

$$U_{p+im} V_{n+im} = \frac{\alpha^{p+n}}{\alpha - \beta} \alpha^{2mi} - \frac{\beta^{p+n}}{\alpha - \beta} \beta^{2mi} + b^n U_{p-n} b^{mi} \ . \qquad (6.1)$$

Let L denote the left-hand side of (4.10). Then, the binomial theorem, using (6.1), gives

$$
\begin{aligned}
L &= \frac{\alpha^{p+n}}{\alpha - \beta}(W_{r+2m} - \alpha^{2m}W_r)^k - \frac{\beta^{p+n}}{\alpha - \beta}(W_{r+2m} - \beta^{2m}W_r)^k \\
&\quad + b^n U_{p-n}(W_{r+2m} - b^m W_r)^k \\
&= b^{rk}U_{2m}^k(\alpha^{p+n-rk}(-1)^k(\alpha - \beta)^k C_2^k - \beta^{p+n-rk}(\alpha - \beta)^k C_1^k)/(\alpha - \beta) \\
&\quad + b^n U_{p-n} Y \; .
\end{aligned} \tag{6.2}
$$

Using (2.1) and (2.2) in (6.2) gives the desired result (4.10).

We obtain (4.11) from (4.10), where $W_n \equiv U_n$. If $W_n \equiv V_n$, then (4.10) gives (4.12) and (4.13), which are also obtained from (6.2), where $C_1 = C_2 = 1$.

7. ADDITIONAL SUMS

Closely related in proof to the above theorems are the following results:

Theorem 5. Let W_n and W_n^* be solutions of (1.1). Let m, p, and n be integers (+, -, or 0). Then

$$
\begin{aligned}
&\sum_{i=0}^{2k}(-1)^i\binom{2k}{i}W_{p+im}W^*_{n-im} \qquad (k > 0) \\
&\qquad = b^p U_m^{2k}(a^2 - 4b)^{k-1}Z_1(m,k) \; ,
\end{aligned} \tag{7.1}
$$

where

$$
\begin{aligned}
&W_0W_1^*V_{n-2jm-p+1} - (bW_0W_0^* + W_1W_1^*)V_{n-2jm-p} \\
&\qquad + bW_0^*W_1V_{n-2jm-p-1} \equiv Z_1(m,j) \; ,
\end{aligned} \tag{7.2}
$$

$$
\begin{aligned}
&\sum_{i=0}^{2k+1}(-1)^i\binom{2k+1}{i}W_{p+im}W^*_{n-im} \qquad (k = 0, 1, \cdots) \\
&\qquad = b^p U_m^{2k+1}(a^2 - 4b)^k \cdot Z_2(m,k),
\end{aligned} \tag{7.3}
$$

where

$$(7.4)\quad \begin{aligned} &W_0 W_1^* U_{n-(2j+1)m-p+1} - (bW_0 W_0^* + W_1 W_1^*) U_{n-(2j+1)m-p} \\ &+ b W_0^* W_1 U_{n-(2j+1)m-p-1} \equiv Z_2(m,j) \quad . \end{aligned}$$

Corollary 5. As special cases of (7.1) and (7.3), we have

$$(7.5)\quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} U_{p+im} U_{n-im} = -b^p U_m^{2k} (a^2 - 4b)^{k-1} V_{n-2km-p} \text{ ,}$$

$$(7.6)\quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{p+im} V_{n-im} = b^p U_m^{2k} (a^2 - 4b)^k V_{n-2km-p} \text{ ,}$$

$$(7.7)\quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} U_{p+im} V_{n-im} = -b^p U_m^{2k} (a^2 - 4b)^k U_{n-2km-p} \text{ ,}$$

$$(7.8)\quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} V_{p+im} U_{n-im} = b^p U_m^{2k} (a^2 - 4b)^k U_{n-2km-p} \text{ ,}$$

$$(7.9)\quad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} U_{p+im} U_{n-im} = -b^p U_m^{2k+1} (a^2 - 4b)^k U_{n-(2k+1)m-p} \text{ ,}$$

$$(7.10)\quad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} V_{p+im} V_{n-im} = b^p U_m^{2k+1} (a^2 - 4b)^{k+1} U_{n-(2k+1)m-p} \text{ ,}$$

$$(7.11)\quad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} U_{p+im} V_{n-im} = -b^p U_m^{2k+1} (a^2 - 4b)^k V_{n-(2k+1)m-p} .$$

Theorem 6. Let W_n and W_n^* be solutions of (1.1). Let m, p, and n be integers (+, -, or 0). Then, for $k = 0, 1, \cdots,$

$$\left(\sum_{i=0}^{k}\binom{k}{i}W_{p+im}W^*_{n-im}\right)(a^2 - 4b) = 2^k Z_3 + b^p V_m^k Z_4 , \tag{7.12}$$

where

$$\begin{aligned} Z_3 &= W_1W_1^* V_{p+n} - b(W_0^* W_1 + W_0W_1^*)V_{p+n-1} + b^2W_0W_0^* V_{p+n-2} \\ &\equiv Z_3(p,n) \end{aligned} \tag{7.13}$$

$$\begin{aligned} Z_4 = W_0 W_1^* V_{n-km-p+1} &- (bW_0W_0^* + W_1W_1^*)V_{n-km-p} \\ &+ bW_0^* W_1 V_{n-km-p-1} . \end{aligned} \tag{7.14}$$

Corollary 6. As special cases of (7.12), we have

$$\begin{aligned} &\left(\sum_{i=0}^{k}\binom{k}{i}U_{p+im}U_{n-im}\right)(a^2 - 4b) \\ &\qquad = 2^k V_{p+n} - b^p V_m^k V_{n-km-p} , \end{aligned} \tag{7.15}$$

$$\sum_{i=0}^{k}\binom{k}{i}V_{p+im}V_{n-im} = 2^k V_{p+n} + b^p V_m^k V_{n-mk-p} , \tag{7.16}$$

$$\sum_{i=0}^{k}\binom{k}{i}U_{p+im}V_{n-im} = 2^k U_{p+n} - b^p V_m^k U_{n-mk-p} . \tag{7.17}$$

Remarks. Special cases of (7.5, (7.6), (7.9), and (7.10) for $U_n \equiv F_n$ and $V_n \equiv L_n$ were given, using matrix methods, in the paper by Hoggatt and Bicknell [3].

8. PROOF OF THEOREMS 5 AND 6

We readily find that

$$(8.1)\qquad \begin{aligned} W_{p+im}W^*_{n-im} &= S_1C_2\alpha^{n-im}\beta^{p+im} + C_1S_2\alpha^{p+im}\beta^{n-im} \\ &\quad + C_1S_1\alpha^{p+n} + C_2S_2\beta^{p+n} \ . \end{aligned}$$

For $r > 0$, we obtain, using the binomial theorem,

$$(8.2)\qquad \begin{aligned} &\sum_{i=0}^{r} (-1)^i \binom{r}{i} W_{p+im}W^*_{n-im} \\ &= S_1C_2\alpha^n\beta^p(1 - \alpha^{-m}\beta^m)^r + C_1S_2\alpha^p\beta^n(1 - \alpha^m\beta^{-m})^r \\ &= b^pU_m^r(\alpha - \beta)^r(S_1C_2\alpha^{n-mr-p} + (-1)^rC_1S_2\beta^{n-mr-p}) \ . \end{aligned}$$

Using (2.1) and (2.2), (8.2) gives (7.1) for $r = 2k$ and (7.3) for $r = 2k + 1$.

Special cases (7.5), $\cdots$, and (7.11) are readily obtained from (7.1) and (7.3) for the choices indicated.

Using (8.1), we readily find that

$$(8.3)\qquad \begin{aligned} \sum_{i=0}^{k} \binom{k}{i} W_{p+im}W^*_{n-im} &= 2^k(C_1S_1\alpha^{p+n} + C_2S_2\beta^{p+n}) \\ &\quad + b^pV_m^k(S_1C_2\alpha^{n-mk-p} + C_1S_2\beta^{n-mk-p}) \ . \end{aligned}$$

Using (2.1) and (2.2), (8.3) reduces to (7.12). Special cases (7.15), $\cdots$, (7.17), are readily obtained from (7.12).

9. MORE SUMS

Introduction of new integer parameters requires that we redefine certain identities by notationally including parameters previously suppressed for simplicity. Thus, we define $Z_1(m, j, p, n)$ by (7.2); $Z_2(m,j,p,n)$ by

(7.4), and $Z_3(p,n)$ by (7.13). Using (8.1), we can obtain the following results, whose lengthy details are omitted.

Theorem 7. Let W_n, W_n^* and W_n^{**} be solutions of (1.1). Let m, p, n, and r be integers (+, -, or 0). Then, we have

$$\begin{aligned}&\sum_{i=0}^{2k}(-1)^i\binom{2k}{i}b^{-mi}W^{**}_{r+2im}W_{p+im}W^*_{n-im} \qquad (k>0)\\ &= (a^2-4b)^{k-1}b^{-2km}U_m^{2k}W^{**}_{r+2km}Z_3(p,n)\\ &\quad + b^{p+r}(a^2-4b)^{k-1}U_{2m}^{2k}[W_0^{**}Z_2(4mk,0,p+r-1,n)\\ &\qquad - W_1^{**}Z_2(4mk,0,p+r,n)],\end{aligned} \tag{9.1}$$

$$\begin{aligned}&\sum_{i=0}^{2k+1}(-1)^i\binom{2k+1}{i}b^{-mi}W^{**}_{r+2im}W_{p+im}W^*_{n-im} \qquad (k\ge 0)\\ &= (a^2-4b)^{k-1}b^{-m(2k+1)}U_m^{k+1}[bW_0^{**}V_{r+m(2k+1)-1}\\ &\qquad - W_1^{**}V_{r+m(2k+1)}]Z_3(p,n)\\ &\quad + b^{p+r}(a^2-4b)^{k-1}U_{2m}^{2k+1}[W_0^{**}Z_1(m,2k+1,p+r-1,n)\\ &\qquad - W_1^{**}Z_1(m,2k+1,p+r,n)],\end{aligned} \tag{9.2}$$

$$\begin{aligned}&(a^2-4b)\sum_{i=0}^{k}\binom{k}{i}b^{-mi}W^{**}_{r+2im}W_{p+im}W^*_{n-im} \qquad (k\ge 0)\\ &= 2^kb^p[W_1^{**}Z_2(0,k,p-r,n) - bW_0^{**}Z_2(0,k,p-r+1,n)]\\ &\quad + b^{p+r}V_{2m}^k[W_0^{**}Z_2(2mk,0,p+r-1,n) - W_1^{**}Z_2(2mk,0,p+r,n)]\\ &\quad + b^{-mk}V_m^kW^{**}_{r+mk}Z_3(p,n).\end{aligned} \tag{9.3}$$

Remarks. As a typical special case, we get from (9.1),

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} b^{-mi} U_{r+2im} U_{p+im} U_{n-im} \qquad (k > 0)$$

(9.4)
$$= (a^2 - 4b)^{k-1} b^{-2km} U_m^{2k} U_{r+2km} V_{p+n}$$
$$+ b^{p+r}(a^2 - 4b)^{k-1} U_{2m}^{2k} U_{n-4mk-p-r} \;.$$

Theorem 8. Let W_n, W_n^*, and W_n^{**} be solutions of (1.1). Let m, p, n, q, and r be integers (+, -, or 0). Then

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} W_{p+im} W_{q+im} W^*_{n-im} W^*_{r-im} \qquad (k > 0)$$

(9.5)
$$= b^p U_m^{2k} (a^2 - 4b)^{k-2} Z_1(m, k, p, n) Z_3(q, r)$$
$$+ b^q U_m^{2k} (a^2 - 4b)^{k-2} Z_1(m, k, q, r) Z_3(p, n)$$
$$+ b^{p+q}(a^2 - 4b)^{k-2} U_{2m}^{2k} A(V_{n+r-p-q-4mk}) \;,$$

where

(9.6)
$$\begin{aligned} A(W_i^{**}) = {} & (W_0 W_1^*)^2 W^{**}_{i+2} - 2W_0 W_1^* (bW_0 W_0^* + W_1 W_1^*) W^{**}_{i+1} \\ & + [b^2(W_0 W_0^*)^2 + 4bW_0 W_1 W_0^* W_1^* + (W_1 W_1^*)^2] W_i^{**} \\ & - 2b W_0^* W_1 (bW_0 W_0^* + W_1 W_1^*) W^{**}_{i-1} + b^2(W_0^* W_1)^2 W^{**}_{i-2} \;. \end{aligned}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} W_{p+im} W_{q+im} W^*_{n-im} W^*_{r-im} \qquad (k \geq 0)$$

(9.7)
$$= b^p U_m^{2k+1} (a^2 - 4b)^{k-1} Z_2(m, k, p, n) Z_3(q, r)$$
$$+ b^q U_m^{2k+1} (a^2 - 4b)^{k-1} Z_2(m, k, q, r) Z_3(p, n)$$
$$+ b^{p+q}(a^2 - 4b)^{k-1} U_{2m}^{2k+1} A(U_{n+r-p-q-2m(2k+1)}) \;.$$

Remarks. As a special case of (9.5), we have

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} U_{p+im} U_{q+im} U_{n-im} U_{r-im} \qquad (k > 0)$$

(9.8)
$$\begin{aligned} &= -b^p U_m^{2k} (a^2 - 4b)^{k-2} V_{n-2km-p} V_{q+r} \\ &\quad - b^q U_m^{2k} (a^2 - 4b)^{k-2} V_{r-2km-q} V_{p+n} \\ &\quad + b^{p+q} (a^2 - 4b)^{k-2} U_{2m}^{2k} V_{n+r-p-q-4mk} . \end{aligned}$$

For Fibonacci, F_n, and Lucas, L_n, sequences, we obtain from (9.8), with $a = -b = 1$,

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} F_{p+im} F_{q+im} F_{n+im} F_{r+im} \qquad (k > 0)$$

(9.9)
$$\begin{aligned} &= (-1)^{r+1} 5^{k-2} F_m^{2k} L_{n+2km+p} L_{q-r} \\ &\quad + (-1)^{n+1} 5^{k-2} F_m^{2k} L_{r+2km+q} L_{p-n} + 5^{k-2} F_{2m}^{2k} L_{n+r+p+q+4mk} . \end{aligned}$$

Theorem 9. Let W_n, W_n^*, and W_n^{**} be solutions of (1.1). Let m, p, n, q, r, and t be integers (+, -, or 0). Then

(9.10)
$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} b^{-mi} W_{t+2im}^{**} W_{p+im} W_{q+im} W_{n-im}^* W_{r-im}^* \qquad (k > 0)$$

$$\begin{aligned} &= b^{p+t} (a^2 - 4b)^{k-2} U_{2m}^{2k} [W_0^{**} Z_2(4mk, 0, p+t-1, n) - W_1^{**} Z_2(4mk, 0, p+t, n)] Z_3(q,r) \\ &\quad + b^{q+t} (a^2 - 4b)^{k-2} U_{2m}^{2k} [W_0^{**} Z_2(4mk, 0, q+t-1, r) - W_1^{**} Z_2(4mk, 0, q+t, r)] Z_3(p,n) \\ &\quad + b^{p+q} (a^2 - 4b)^{k-2} U_m^{2k} [W_1^{**} A(U_{n+r+t-p-q-2mk}) - bW_0^{**} A(U_{n+r+t-p-q-2mk-1})] \\ &\quad + b^{p+q+t} (a^2 - 4b)^{k-2} U_{3m}^{2k} [W_0^{**} A(U_{n+r-p-q-t-6mk+1}) - W_1^{**} A(U_{n+r-p-q-t-6mk})] \\ &\quad + b^{-2mk} (a^2 - 4b)^{k-2} U_m^{2k} W_{t+2mk}^{**} Z_3(p,n) Z_3(q,r) \\ &\quad + b^{-2mk+n+q} (a^2 - 4b)^{k-2} U_m^{2k} W_{t+2mk}^{**} V_{p+r-n-q} D(W_0, W_1) D(W_0^*, W_1^*) , \end{aligned}$$

where

$$D(W_0, W_1) = -(W_1^2 - aW_0W_1 + bW_0^2) \quad . \tag{9.11}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} b^{-mi} W^{**}_{t+2im} W_{p+im} W_{q+im} W^*_{n-im} W^*_{r-im} \qquad (k \geq 0)$$

$$\begin{aligned}
&= b^{p+t}(a^2 - 4b)^{k-2} U_{2m}^{2k+1} [W_0^{**} Z_1(m, 2k+1, p+t-1, n) \\
&\qquad - W_1^{**} Z_1(m, 2k+1, p+t, n)] Z_3(q, r) \\
&+ b^{q+t}(a^2 - 4b)^{k-2} U_{2m}^{2k+1} [W_0^{**} Z_1(m, 2k+1, q+t-1, r) \\
&\qquad - W_1^{**} Z_1(m, 2k+1, q+t, r)] Z_3(p, n) \\
&+ b^{p+q}(a^2 - 4b)^{k-2} U_m^{2k+1} [W_1^{**} A(V_{n+r+t-p-q-m(2k+1)}) \\
&\qquad - bW_0^{**} A(V_{n+r+t-p-q-m(2k+1)-1})] \\
&+ b^{p+q+t}(a^2 - 4b)^{k-2} U_{3m}^{2k+1} [W_0^{**} A(V_{n+r-p-q-t-3m(2k+1)+1}) \\
&\qquad - W_1^{**} A(V_{n+r-p-q-t-3m(2k+1)})] \\
&+ b^{-m(2k+1)}(a^2 - 4b)^{k-2} U_m^{2k+1} [bW_0^{**} V_{t+m(2k+1)-1} \\
&\qquad - W_1^{**} V_{t+m(2k+1)}] Z_3(p, n) Z_3(q, r) \\
&+ b^{-m(2k+1)+n+q}(a^2 - 4b)^{k-2} U_m^{2k+1} (bW_0^{**} V_{t+m(2k+1)-1} \\
&\qquad - W_1^{**} V_{t+m(2k+1)}) D(W_0^*, W_1^*) D(W_0, W_1) V_{p+r-n-q} .
\end{aligned} \tag{9.12}$$

Theorem 10. Let W_n and W^*_n be solutions of (1.1). Let m, p, n, q, r, t, and s be integers (+, -, or 0). Then, for $k > 0$,

$$\begin{aligned}
&\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} W_{p+im} W_{q+im} W_{s+im} W^*_{n-im} W^*_{r-im} W^*_{t-im} \\
&= b^s U_m^{2k} (a^2 - 4b)^{k-3} Z_1(m, k, s, t) Z_3(p, n) Z_3(q, r) \\
&+ b^p U_m^{2k} (a^2 - 4b)^{k-3} Z_1(m, k, p, n) Z_3(q, r) Z_3(s, t) \\
&+ b^q U_m^{2k} (a^2 - 4b)^{k-3} Z_1(m, k, q, r) Z_3(p, n) Z_3(s, t) \\
&+ b^{p+q} U_{2m}^{2k} (a^2 - 4b)^{k-3} Z_3(s, t) A(V_{n+r-p-q-4mk}) \\
&+ b^{p+s} U_{2m}^{2k} (a^2 - 4b)^{k-3} Z_3(q, r) A(V_{n+t-p-s-4mk}) \\
&+ b^{q+s} U_{2m}^{2k} (a^2 - 4b)^{k-3} Z_3(p, n) A(V_{r+t-q-s-4mk})
\end{aligned} \tag{9.13}$$

(Continued, next page.)

$$+ b^{p+q+t}U_m^{2k}(a^2-4b)^{k-3}D(W_0,W_1)D(W_0^*,W_1^*)Z_1(m,k,p+q+t-r-s,n)$$
$$+ b^{n+q+s}U_m^{2k}(a^2-4b)^{k-3}D(W_0,W_1)D(W_0^*,W_1^*)Z_1(m,k,s,t)V_{p+r-n-q}$$
$$+ W_0W_1^*b^{p+q+s}U_{3m}^{2k}(a^2-4b)^{k-3}A(V_{n+r+t-p-q-s-6mk+1})$$
$$- (bW_0W_0^* + W_1W_1^*)b^{p+q+s}U_{3m}^{2k}(a^2-4b)^{k-3}A(V_{n+r+t-p-q-s-6mk})$$
$$+ W_0^*W_1b^{p+q+s+1}U_{3m}^{2k}(a^2-4b)^{k-3}A(V_{n+r+t-p-q-s-6mk-1}),$$

(9.14)

$$\sum_{i=0}^{2k+1}(-1)^i\binom{2k+1}{i}W_{p+im}W_{q+im}W_{s+im}W_{n-im}^*W_{r-im}^*W_{t-im}^* \qquad (k \ge 0)$$
$$= b^sU_m^{2k+1}(a^2-4b)^{k-2}Z_2(m,k,s,t)Z_3(p,n)Z_3(q,r)$$
$$+ b^pU_m^{2k+1}(a^2-4b)^{k-2}Z_2(m,k,p,n)Z_3(q,r)Z_3(s,t)$$
$$+ b^qU_m^{2k+1}(a^2-4b)^{k-2}Z_2(m,k,q,r)Z_3(p,n)Z_3(s,t)$$
$$+ b^{p+q}U_{2m}^{2k+1}(a^2-4b)^{k-2}Z_3(s,t)A(U_{n+r-p-q-2m(2k+1)})$$
$$+ b^{p+s}U_{2m}^{2k+1}(a^2-4b)^{k-2}Z_3(q,r)A(U_{n+t-p-s-2m(2k+1)})$$
$$+ b^{q+s}U_{2m}^{2k+1}(a^2-4b)^{k-2}Z_3(p,n)A(U_{r+t-q-s-2m(2k+1)})$$
$$+ b^{p+q+t}U_m^{2k+1}(a^2-4b)^{k-2}D(W_0,W_1)D(W_0^*,W_1^*)Z_2(m,k,p+q+t-r-s,n)$$
$$+ b^{n+q+s}U_m^{2k+1}(a^2-4b)^{k-2}D(W_0,W_1)D(W_0^*,W_1^*)Z_2(m,k,s,t)V_{p+r-n-q}$$
$$+ W_0W_1^*b^{p+q+s}U_{3m}^{2k+1}(a^2-4b)^{k-2}A(U_{n+r+t-p-q-s+1-3m(2k+1)})$$
$$- (bW_0W_0^* + W_1W_1^*)b^{p+q+s}U_{3m}^{2k+1}(a^2-4b)^{k-2}A(U_{n+r+t-p-q-s-3m(2k+1)})$$
$$+ W_0^*W_1b^{p+q+s+1}U_{3m}^{2k+1}(a^2-4b)^{k-2}A(U_{n+r+t-p-q-s-1-3m(2k+1)}).$$

Remarks. As a special case of (9.13), we have

(9.15)

$$\sum_{i=0}^{2k}(-1)^i\binom{2k}{i}U_{p+im}U_{q+im}U_{s+im}U_{n-im}U_{r-im}U_{t-im} \qquad (k > 0)$$
$$= -b^sU_m^{2k}(a^2-4b)^{k-3}V_{t-2km-s}V_{p+n}V_{q+r}$$
$$- b^pU_m^{2k}(a^2-4b)^{k-3}V_{n-2km-p}V_{q+r}V_{s+t}$$
$$- b^qU_m^{2k}(a^2-4b)^{k-3}V_{r-2km-q}V_{p+n}V_{s+t}$$

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$$+ b^{p+q} U_{2m}^{2k} (a^2 - 4b)^{k-3} V_{s+t} V_{n+r-p-q-4mk}$$

$$+ b^{p+s} U_{2m}^{2k} (a^2 - 4b)^{k-3} V_{q+r} V_{n+t-p-s-4mk}$$

$$+ b^{q+s} U_{2m}^{2k} (a^2 - 4b)^{k-3} V_{p+n} V_{r+t-q-s-4mk}$$

$$- b^{p+q+t} U_{m}^{2k} (a^2 - 4b)^{k-3} V_{n-2km-p-q-t+r+s}$$

$$- b^{n+q+s} U_{m}^{2k} (a^2 - 4b)^{k-3} V_{t-2km-s} V_{p+r-n-q}$$

$$- b^{p+q+s} U_{3m}^{2k} (a^2 - 4b)^{k-3} V_{n+r+t-p-q-s-6mk} \; .$$

10. BINOMIAL SUMS WITH TRIPLE CROSS-PRODUCTS

The following results are an extension of Theorem 3.

Theorem 11. Let W_n be a solution of (1.1). Let $m, p_1, p_2,$ and p_3 be integers (+, -, or 0). Let $p_1 + p_2 + p_3 = p$. Then

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i \left(\prod_{j=1}^{3} W_{p_j+im} \right) \qquad (k > 0)$$

$$(10.1) \qquad = U_{3m}^{2k} (a^2 - 4b)^{k-1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i U_{p+2mk-i}$$

$$+ D(W_0, W_1) b^{4mk} U_m^{2k} (a^2 - 4b)^{k-1} \sum_{j=1}^{3} b^{p_j} W_{p-2p_j-2mk} \; ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i \left(\prod_{j=1}^{3} W_{p_j+im} \right) \qquad (k \geq 0)$$

$$(10.2) \qquad = -U_{3m}^{2k+1} (a^2 - 4b)^{k-1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i V_{p+m(2k+1)-i}$$

$$- D(W_0, W_1) b^{2m(2k+1)} U_m^{2k+1} (a^2 - 4b)^{k-1} Y_1 \qquad ,$$

where

$$(10.3)\qquad Y_1 = \sum_{j=1}^{3} b^{p_j}(W_1 V_{p-2p_j-m(2k+1)} - bW_0 V_{p-2p_j-m(2k+1)-1}) \ .$$

Special cases of Theorem 11, with $p_1 = p_2 = p_3 = n$, are given by

$$(10.4)\qquad \begin{aligned} &\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i U_{n+im}^3 \qquad (k > 0) \\ &= (a^2 - 4b)^{k-1} (U_{3m}^{2k} U_{3n+2mk} - 3b^{n+4mk} U_m^{2k} U_{n-2mk}) \ , \end{aligned}$$

$$(10.5)\qquad \begin{aligned} &\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i V_{n+im}^3 \qquad (k > 0) \\ &= (a^2 - 4b)^{k} (U_{3m}^{2k} V_{3n+2mk} + 3b^{n+4mk} U_m^{2k} V_{n-2mk}) \ , \end{aligned}$$

$$(10.6)\qquad \begin{aligned} &\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i U_{n+im}^3 \qquad (k \geq 0) \\ &= (a^2 - 4b)^{k-1} (-U_{3m}^{2k+1} V_{3n+m(2k+1)} + 3b^{n+2m(2k+1)} U_m^{2k+1} V_{n-m(2k+1)}) \ , \end{aligned}$$

$$(10.7)\qquad \begin{aligned} &\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i V_{n+im}^3 \qquad (k \geq 0) \\ &= (a^2 - 4b)^{k+1} (-U_{3m}^{2k+1} U_{3n+m(2k+1)} - 3b^{n+2m(2k+1)} U_m^{2k+1} U_{n-m(2k+1)}) \ . \end{aligned}$$

11. BINOMIAL SUMS WITH FOUR CROSS-PRODUCTS

The following results are an extension of Theorem 3.

Theorem 12. Let W_n be a solution of (1.1). Let m, p_1, p_2, p_3, and p_4 be integers (+, -, or 0). Let $p_1 + p_2 + p_3 + p_4 = p$. Then

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i \left(\prod_{j=1}^{4} W_{p_j+im} \right) \qquad (k > 0) \tag{11.1}$$

$$= U_{4m}^{2k} (a^2 - 4b)^{k-2} \sum_{i=0}^{4} (-1)^i \binom{4}{i} W_1^{4-i} (b\, W_0)^i\, V_{p+2mk-i}$$

$$+ U_m^{2k} U_{2m}^{2k} D^2(W_0, W_1)(a^2 - 4b)^{2k-2} \sum_{i=1}^{3} b^{2mk+p_1+p_{1+i}}\, V_{p-2p_1-2p_{1+i}}$$

$$+ U_{2m}^{2k} D(W_0, W_1)(a^2 - 4b)^{k-2} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i \sum_{j=1}^{4} b^{4mk+p_j} V_{p-2p_j-2mk-i},$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i \left(\prod_{j=1}^{4} W_{p_j+im} \right) \qquad (k \geq 0) \tag{11.2}$$

$$= -U_{4m}^{2k+1} (a^2 - 4b)^{k-1} \sum_{i=0}^{4} (-1)^i \binom{4}{i} W_1^{4-i} (bW_0)^i\, U_{p+m(2k+1)-i}$$

$$+ (U_m U_{2m})^{2k+1} D^2(W_0, W_1)(a^2 - 4b)^{2k-1} \sum_{i=1}^{3} b^{m(2k+1)+p_1+p_{1+i}} V_{p-2p_1-2p_{1+i}}$$

$$- U_{2m}^{2k+1} D(W_0, W_1)(a^2 - 4b)^{k-1}$$

$$\cdot \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i \sum_{j=1}^{4} b^{2m(2k+1)+p_j} U_{p-2p_j-m(2k+1)-i}.$$

Special cases of Theorem 12, with $p_1 = p_2 = p_3 = p_4 = n$, are given by

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i U_{n+im}^4 \qquad (k > 0)$$

$$(11.3)\qquad \begin{aligned} &= U_{4m}^{2k}(a^2 - 4b)^{k-2} V_{4n+2mk} \\ &+ 6\, U_m^{2k} U_{2m}^{2k} (a^2 - 4b)^{2k-2}\, b^{2mk+2n} \\ &- 4\, U_{2m}^{2k} (a^2 - 4b)^{k-2}\, b^{4mk+n} V_{2n-2mk} \quad , \end{aligned}$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i V_{n+im}^4 \qquad (k > 0)$$

$$(11.4)\qquad \begin{aligned} &= U_{4m}^{2k} (a^2 - 4b)^k V_{4n+2mk} \\ &+ 6\, U_{2m}^{2k} (a^2 - 4b)^{2k}\, b^{2mk+2n} U_m^{2k} \\ &+ 4\, U_{2m}^{2k} (a^2 - 4b)^k\, b^{4mk+n} V_{2n-2mk} \quad , \end{aligned}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i U_{n+im}^4 \qquad (k \geq 0)$$

$$(11.5)\qquad \begin{aligned} &= -U_{4m}^{2k+1} (a^2 - 4b)^{k-1} U_{4n+m(2k+1)} \\ &+ 6(U_m U_{2m})^{2k+1} (a^2 - 4b)^{2k-1}\, b^{m(2k+1)+2n} \\ &+ 4\, U_{2m}^{2k+1} (a^2 - 4b)^{k-1}\, b^{2m(2k+1)+n} U_{2n-m(2k+1)} \quad , \end{aligned}$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i V_{n+im}^4 \qquad (k \geq 0)$$

$$(11.6)\qquad \begin{aligned} &= -U_{4m}^{2k+1} (a^2 - 4b)^{k+1} U_{4n+m(2k+1)} \\ &+ 6(U_m U_{2m})^{2k+1} (a^2 - 4b)^{2k+1}\, b^{m(2k+1)+2n} \\ &- 4\, U_{2m}^{2k+1} (a^2 - 4b)^{k+1}\, b^{2m(2k+1)+n} U_{2n-m(2k+1)} \quad . \end{aligned}$$

12. BINOMIAL SUMS WITH MIXED CROSS-PRODUCTS

The following results are companion results for Theorems 11 and 12.

Theorem 13. Let W_n be a solution of (1.1). Let m, p_1, p_2, and p_3 be integers (+, -, or 0). Let $p_1 + p_2 + p_3 = p$. Then

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i \left(V_{p_1+im} \prod_{j=2}^{3} W_{p_j+im} \right) \qquad (k > 0)$$

(12.1)

$$= U_{3m}^{2k} (a^2 - 4b)^{k-1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p+2mk-i}$$

$$+ D(W_0, W_1) b^{4mk} U_m^{2k} (a^2 - 4b)^{k-1} \sum_{j=2}^{3} b^{p_j} V_{p-2p_j-2mk}$$

$$+ U_m^{2k} (a^2 - 4b)^{k-1} b^{4mk+p_1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p-2p_1-2mk-i} ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i \left(V_{p_1+im} \prod_{j=2}^{3} W_{p_j+im} \right) \qquad (k \geq 0)$$

(12.2)

$$= -U_{3m}^{2k+1} (a^2 - 4b)^k \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i U_{p+m(2k+1)-i}$$

$$- D(W_0, W_1) b^{2m(2k+1)} U_m^{2k+1} (a^2 - 4b)^k \sum_{j=2}^{3} b^{p_j} U_{p-2p_j-m(2k+1)}$$

$$- U_m^{2k+1} (a^2 - 4b)^k b^{2m(2k+1)+p_1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i U_{p-2p_1-m(2k+1)-i}$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{3} W_{p_j+im} \right) \qquad (k > 0)$$

(12.3)

$$= U_{3m}^{2k} (a^2 - 4b)^{k-1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i U_{p+2mk-i}$$

$$+ U_m^{2k} D(W_0, W_1)(a^2 - 4b)^{k-1} \sum_{j=2}^{3} b^{4mk+p_j} U_{p-2p_j-2mk}$$

$$- (a^2 - 4b)^{k-1} U_m^{2k} b^{4mk+p_1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i U_{p-2p_1-2mk-i} \, ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{3} V_{p_j+im} \right) \qquad (k > 0)$$

(12.4)

$$= (a^2 - 4b)^k U_{3m}^{2k} U_{p+2mk}$$

$$+ U_m^{2k} (a^2 - 4b)^k \sum_{j=2}^{3} b^{4mk+p_j} U_{p-2p_j-2mk}$$

$$- (a^2 - 4b)^k U_m^{2k} b^{4mk+p_1} U_{p-2p_1-2mk} \, ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{3} W_{p_j+im} \right) \qquad (k \geq 0)$$

(12.5)

$$= -U_{3m}^{2k+1} (a^2 - 4b)^{k-1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^i (bW_0)^i V_{p+m(2k+1)-i}$$

$$- U_m^{2k+1} D(W_0, W_1)(a^2 - 4b)^{k-1} \sum_{j=2}^{3} b^{2m(2k+1)+p_j} V_{p-2p_j-m(2k+1)}$$

$$+ (a^2 - 4b)^{k-1} U_m^{2k+1} b^{2m(2k+1)+p_1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p-2p_1-m(2k+1)-i} \, ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{3} V_{p_j+im} \right) \qquad (k \geq 0)$$

$$= -U_{3m}^{2k+1} (a^2 - 4b)^k V_{p+m(2k+1)}$$

(12.6)

$$- U_m^{2k+1} (a^2 - 4b)^k \sum_{j=2}^{3} b^{2m(2k+1)+p_j} V_{p-2p_j-m(2k+1)}$$

$$+ (a^2 - 4b)^k U_m^{2k+1} b^{2m(2k+1)+p_1} V_{p-2p_1-m(2k+1)} \quad .$$

Theorem 14. Let W_n be a solution of (1.1). Let m, p_1, p_2, p_3, and p_4 be integers (+, -, or 0).

Let $p_1 + p_2 + p_3 + p_4 = p$. Then

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i \left(V_{p_1+im} \prod_{j=2}^{4} W_{p_j+im} \right) \qquad (k > 0)$$

$$= (a^2 - 4b)^{k-1} U_{4m}^{2k} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i U_{p+2mk-i}$$

(12.7)
$$+ D(W_0, W_1)(a^2 - 4b)^{2k-1} (U_m U_{2m})^{2k} \sum_{i=1}^{3} b^{2mk+p_1+p_{1+i}} W_{p-2p_1-2p_{1+i}}$$

$$+ D(W_0, W_1)(a^2 - 4b)^{k-1} U_{2m}^{2k} \sum_{j=2}^{4} b^{4mk+p_j} W_{p-2p_j-2mk}$$

$$+ (a^2 - 4b)^{k-1} U_{2m}^{2k} b^{4mk+p_1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i U_{p-2p_1-2mk-i} \,,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i \left(V_{p_1+im} \prod_{j=2}^{4} W_{p_j+im} \right) \qquad (k \geq 0)$$

$$= -(a^2 - 4b)^{k-1} U_{4m}^{2k+1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i V_{p+m(2k+1)-i}$$

(12.8)

$$+ D(W_0, W_1)(a^2 - 4b)^{2k} (U_m U_{2m})^{2k+1} \sum_{i=1}^{3} b^{m(2k+1)+p_1+p_{1+i}} W_{p-2p_1-2p_{1+i}}$$

$$- (a^2 - 4b)^{k-1} U_{2m}^{2k+1} b^{2m(2k+1)+p_1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i V_{p-2p_1-m(2k+1)-i}$$

$$- (a^2 - 4b)^{k-1} D(W_0, W_1) U_{2m}^{2k+1} b^{2m(2k+1)} Y_2(2k+1) \quad ,$$

where

(12.9) $$Y_2(k) = \sum_{j=2}^{4} b^{p_j} (W_1 V_{p-2p_j-mk} - bW_0 V_{p-2p_j-mk-1}) \quad ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i \left(\prod_{j=1}^{2} V_{p_j+im} \right) \left(\prod_{j=3}^{4} W_{p_j+im} \right) \qquad (k > 0)$$

(12.10)

$$= (a^2 - 4b)^{k-1} U_{4m}^{2k} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p+2mk-i}$$

$$+ (a^2 + 4b)^{2k-1} (U_m U_{2m})^{2k} b^{2mk+p_1+p_2} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p-2p_1-2p_2-i}$$

$$+ D(W_0, W_1)(a^2 - 4b)^{2k-1} (U_m U_{2m})^{2k} \sum_{j=1}^{2} b^{2mk+p_1+p_{2+j}} V_{p-2p_1-2p_{2+j}}$$

(Continued, next page.)

$$+ (a^2 - 4b)^{k-1} U_{2m}^{2k} \sum_{j=1}^{2} b^{4mk+p_j} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p-2p_j-2mk-i}$$

$$+ (a^2 - 4b)^{k-1} D(W_0, W_1) U_{2m}^{2k} \sum_{j=3}^{4} b^{4mk+p_j} V_{p-2p_j-2mk} \quad ,$$

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i \left(\prod_{j=1}^{2} V_{p_j+im} \right) \left(\prod_{j=3}^{4} W_{p_j+im} \right) \tag{12.11}$$

$(k \geq 0)$

$$= -(a^2 - 4b)^k U_{4m}^{2k+1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i U_{p+m(2k+1)-i}$$

$$+ (a^2 - 4b)^{2k} (U_m U_{2m})^{2k+1} b^{m(2k+1)+p_1+p_2} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i V_{p-2p_1-2p_2-i}$$

$$+ D(W_0, W_1)(a^2 - 4b)^{2k} (U_m U_{2m})^{2k+1} \sum_{j=1}^{2} b^{m(2k+1)+p_1+p_{2+j}} V_{p-2p_1-2p_{2+j}}$$

$$- (a^2 - 4b)^k U_{2m}^{2k+1} \sum_{j=1}^{2} b^{2m(2k+1)+p_j} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (bW_0)^i U_{p-2p_j-m(2k+1)-i}$$

$$- (a^2 - 4b)^k U_{2m}^{2k+1} D(W_0, W_1) \sum_{j=3}^{4} b^{2m(2k+1)+p_j} U_{p-2p_j-m(2k+1)} \quad ,$$

$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{4} W_{p_j+im} \right) \tag{12.12}$$

$(k > 0)$

$$= (a^2 - 4b)^{k-2} U_{4m}^{2k} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i V_{p+2mk-i}$$

(Continued, next page.)

$$- (a^2 - 4b)^{2k-2} D(W_0, W_1)(U_m U_{2m})^{2k} \sum_{j=1}^{3} b^{2mk+p_1+p_{1+j}} (W_1 V_{p-2p_1-2p_{1+j}} - bW_0 V_{p-2p_1-2p_{1+j}-1})$$

$$- (a^2 - 4b)^{k-2} U_{2m}^{2k} b^{4mk+p_1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i V_{p-2p_1-2mk-i}$$

$$+ (a^2 - 4b)^{k-2} U_{2m}^{2k} D(W_0, W_1) \sum_{j=2}^{4} b^{4mk+p_j} (W_1 V_{p-2p_j-2mk} - bW_0 V_{p-2p_j-2mk-1}) ,$$

$$(12.13) \quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{3m})^{2k-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{4} V_{p_j+im} \right) \qquad (k > 0)$$

$$= (a^2 - 4b)^k U_{4m}^{2k} U_{p+2mk}$$

$$- (a^2 - 4b)^{2k} U_m^{2k} U_{2m}^{2k} \sum_{j=1}^{3} b^{2mk+p_1+p_{1+j}} U_{p-2p_1-2p_{1+j}}$$

$$- (a^2 - 4b)^k U_m^{2k} b^{4mk+p_1} U_{p-2p_1-2mk}$$

$$+ (a^2 - 4b)^k U_{2m}^{2k} \sum_{j=2}^{4} b^{4mk+p_j} U_{p-2p_j-2mk} ,$$

$$(12.14) \quad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{4} W_{p_j+im} \right) \qquad (k \geq 0)$$

$$= - (a^2 - 4b)^{k-1} U_{4m}^{2k+1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i U_{p+m(2k+1)-i}$$

$$- (a^2 - 4b)^{2k-1} D(W_0, W_1)(U_m U_{2m})^{2k+1} \sum_{j=1}^{3} b^{m(2k+1)+p_1+p_{1+j}} (W_1 V_{p-2p_1-2p_{1+j}} - bW_0 V_{p-2p_1-2p_{1+j}-1})$$

(Continued, next page.)

$$- (a^2 - 4b)^{k-1} D(W_0, W_1) U_{2m}^{2k+1} \sum_{j=2}^{4} b^{2m(2k+1)+p_j} W_{p-2p_j-m(2k+1)}$$

$$+ (a^2 - 4b)^{k-1} U_{2m}^{2k+1} b^{2m(2k+1)+p_1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i U_{p-2p_1-m(2k+1)-i} ,$$

$$(12.15) \quad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_m^i \left(U_{p_1+im} \prod_{j=2}^{4} V_{p_j+im} \right) \quad (k > 0)$$

$$= -(a^2 - 4b)^k U_{4m}^{2k+1} V_{p+m(2k+1)}$$

$$- (a^2 - 4b)^{2k+1} (U_m U_{2m})^{2k+1} \sum_{j=1}^{3} b^{m(2k+1)+p_1+p_{1+j}} U_{p-2p_1-2p_{1+j}}$$

$$- (a^2 - 4b)^k U_{2m}^{2k+1} \sum_{j=2}^{4} b^{2m(2k+1)+p_j} V_{p-2p_j-m(2k+1)}$$

$$+ (a^2 - 4b)^k U_{2m}^{2k+1} b^{2m(2k+1)+p_1} V_{p-2p_1-m(2k+1)} .$$

13. REMARKS ON THE PAPER BY CARLITZ AND FERNS [4]

All the important identities of Sections 1 and 2 of the above paper are special cases of our general results. Indeed, for the proper choices of parameters, our result, (1.7), contains as special cases, identities (1.6), (1.8), (1.10), (1.11), and (1.12) of [4, pp. 62-64]. In [4, pp. 65-66], I noted misprints and omissions in (2.9) (for n odd), (2.10) (for n even and odd), and (2.11) (for n even and odd). If these errors are corrected, we can then say, for the proper choices of parameters, our (1.9) gives their (2.8) and (2.10) for even n; our (1.13) gives their (2.8) and (2.10) for odd n. Also, our (1.12) gives their (2.9) and (2.11) for odd n. The odd and even n refers to their identity usage, not ours. Our (1.11) gives their (2.9) and (2.11) for even n.

The remaining portion of [4] obtained transformation identities for the Fibonacci and Lucas sequences as an application of Legendre and Jacobi polynomials. Under proper linear substitutions, these same polynomials could give transformation identities for U_n and V_n of (1.1). We will illustrate these ideas with a pair of identities suitable for our purposes.

The following pair of polynomial identities,

$$x^{2n+1} - (x-1)^{2n+1} = \sum_{i=0}^{n} \frac{2n+1}{2i+1} \binom{n+i}{2i} (x^2 - x)^{n-i} , \tag{13.1}$$

$$x^{2n+2} - (x-1)^{2n+2} = (2x-1) \sum_{i=0}^{n} \binom{n+1+i}{2i+1} (x^2 - x)^{n-i} , \tag{13.2}$$

appeared as a proposed problem 4356, p. 479, in the American Mathematical Monthly, 56 (1949), and their solution, in the same journal, 58 (1951), appears on pp. 268-269. We now proceed to apply (13.1) and (13.2) to obtain identities for U_n and V_n of (1.1).

Recalling that α and β are roots of $x^2 = ax - b$ (see (1.1)), set $x = (a/b)y$ in (13.1) to obtain

$$a^{2n+1}y^{2n+1} - (ay-b)^{2n+1} = \sum_{i=0}^{n} \frac{2n+1}{2i+1} \binom{n+i}{2i} b^{2i+1} [ay(ay-b)]^{n-i} . \tag{13.3}$$

Thus, (13.3) for $y = \alpha$ and $y = \beta$ gives the identities

$$a^{2n+1}V_{2n+1} - V_{4n+2} = \sum_{i=0}^{n} \frac{2n+1}{2i+1} \binom{n+i}{2i} b^{2i+1} a^{n-i} V_{3n-3i} , \tag{13.4}$$

$$a^{2n+1}U_{2n+1} - U_{4n+2} = \sum_{i=0}^{n} \frac{2n+1}{2i+1} \binom{n+i}{2i} b^{2i+1} a^{n-i} U_{3n-3i} . \tag{13.5}$$

In (13.2), set $x = (ay)/(2b)$ to obtain for even $n = 2k$, noting that $a\alpha - 2b = \alpha^2 - b = \alpha(\alpha - \beta)$, $\alpha\beta - 2b = \beta(\beta - \alpha)$,

$$\left\{a^{4k+2} - (a^2 - 4b)^{2k+1} - (a^2 - 4b)^k a^{2k}(4b)(2k + 1)\right\} V_{4k+2}$$

$$(13.6) \qquad = 2\sum_{j=0}^{k-1} (a^2 - 4b)^j a^{2j} (2b)^{4(k-j)+1} \binom{4k + 1 - 2j}{2j} V_{4j+2}$$

$$+ 2\sum_{i=1}^{k} (a^2 - 4b)^i a^{2i-1} (2b)^{4(k-i)+3} \binom{4k + 2 - 2i}{2i - 1} U_{4i} \quad .$$

An identity similar to (13.6) is obtained for $n = 2k + 1$. We note that the factor $(2x - 1)$ in (13.2) is troublesome for obtaining identities in U_n and V_n for (1.1), but is not so for the Fibonacci sequences.

Additional identities for U_n and V_n are readily obtained from (13.1). For complete generality, we note that α and β satisfy $x^m = U_m x - bU_{m-1}$. Thus, from (13.1), we obtain, having set $x = (U_m y)/(bU_{m-1})$, the general identity

$$(13.7) \qquad U_m^{2n+1} W_{2n+1+p} - W_{2mn+m+p}$$

$$= \sum_{i=0}^{n} \frac{2n + 1}{2i + 1} \binom{n + i}{2i} U_m^{n-i} (bU_{m-1})^{2i+1} W_{(m+1)(n-i)+p} \quad ,$$

where U_n and W_n are solutions of (1.1).

It should be noted that (13.1) gives Fibonacci identities that are not special cases of (13.7). As a partial listing, we have

$$(13.8) \qquad L_{2n+1} = \sum_{i=0}^{n} \frac{2n + 1}{2i + 1} \binom{n + i}{2i} \quad ,$$

$$(13.9) \qquad H_{12n+6+p} - 4^{2n+1} H_{6n+3+p} = \sum_{i=0}^{n} \frac{2n + 1}{2i + 1} \binom{n + i}{2i} 4^{n-i} H_{9n-9i+p} \quad ,$$

[Continued on page 421.]