# Abstracts

# The Édouard Lucas Memorial Lecture

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**Hugh C. Williams**, University of Calgary, Calgary, AB, Canada
*Mersenne, Fibonacci and Lucas:*
*The Mersenne Prime Story and Beyond*

On Dec. 26 of last year, it was announced that the 50th known Mersenne prime had been identified. This is an enormous number of 23,249,425 decimal digits and is currently the largest known prime number. In spite of the size of this number we are able to prove it prime by a simple algorithm that was discovered in 1876 by Édouard Lucas. Lucas discovered this procedure as a result of his examination of the properties of Fibonacci numbers.

In this talk I will briefly discuss the development of the concept of a Mersenne prime and then describe Lucas' ideas concerning how the primality of such numbers can be established. I will also detail some aspects of Lucas' career and conclude with a discussion of his unsuccessful search for a generalization of his technique.

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# Abstracts – Contributed Talks

Abstracts are listed in alphabetical order by speaker.

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**Sadjia Abbad**, Saad Dahlab University, Blida, Algeria

*Companion Sequences Associated to the $r$-Fibonacci Sequence*

In this talk, we define the $r$-Lucas sequences of type $s$. These sequences constitute a family of companion sequences of the generalized $r$- Fibonacci sequences. We establish the corresponding Binet formula and evaluate generating functions. Therefore we extend the definition of $V_n^{(r,s)}$ to negative $n$. Also, we exhibit some convolution relations which generalize some known identities such as Cassinis.

(Joint work with Hacène Belbachir.)

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**Michael A. Allen**, Mahidol University, Bangkok, Thailand

*A New Combinatorial Interpretation of the Fibonacci Numbers Squared*

We consider the tiling of an $n$-board (a $1 \times n$ array of square cells of unit width) with half-squares ($\frac{1}{2} \times 1$ tiles) and $(\frac{1}{2}, \frac{1}{2})$-fence tiles. A $(\frac{1}{2}, \frac{1}{2})$-fence tile is composed of two half-squares separated by a gap of width $\frac{1}{2}$. We show that the number of ways to tile an $n$-board using these types of tiles equals $F_{n+1}^2$ where $F_n$ is the $n$th Fibonacci number. We use these tilings to devise combinatorial proofs of identities relating the Fibonacci numbers squared to one another. Some of these identities appear to be new.

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**Abdullah Al-Shaghay**, Dalhousie University, Halifax, NS, Canada

*Irreducibility and Roots of a Class of Polynomials*

In 2012 Harrington studied the factorization of trinomials of the form $x^n + cx^{n-1} + d \in \mathbb{Z}[x]$. As an application of these results on trinomials, he proves factorization properties of polynomials of the form $x^n + c(x^{n-1} + \ldots + x + 1) \in \mathbb{Z}[x]$. In this presentation, results regarding the factorization and roots of polynomials of the form $x^n + c(x^{n-a-1} + \ldots + x + 1) \in \mathbb{Z}[x]$ are introduced. Analogously to Harrington, quadrinomials of the form $x^{n+1} \pm x^n \pm cx^{n-a} \pm c$ associated to our polynomials are conisdered.

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**Peter G. Anderson**, RIT, Rochester, NY, USA

*More Remarkable Continued Functions*

Some sequences of linear functions of the form $T_n = \frac{x+b_n}{c_n}$ which satisfies a Fibonacci-like composition rule, $T_{n+1} = T_n \circ T_{n-1}$ has a sequence of fixed points $f_n = b_n/(c_n - 1)$ involving a remarkable continued fraction $[a_0, a_1, \ldots]$ (either finite for any $T_n$ or infinite for $\lim_{n\to\infty} T_n$) satisfying a Fibonacci-like multiplication rule, $a_{m+1} = a_m a_{m-1}$, for $m \geq 1$.

For every pair of positive integers, $a_0, a_1$, there is a pair of functions, $T_0, T_1$, giving the sequence $T_n$, as above, with fixed points involving the continued fraction as described above.

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**Christian Ballot**, University of Caen, Caen, France

*Variations on Catalan Lucasnomials*

If $U = (U_n)_{n\geq 0}$ is a sequence of nonzero integers, then one may consider the generalized binomial coefficients, $\binom{m}{n}_U$, with respect to $U$. They are defined for $m \geq n \geq 0$ as follows

$$\binom{m}{n}_U = \frac{U_m U_{m-1} \ldots U_{m-n+1}}{U_n U_{n-1} \ldots U_1},$$

if $m \geq n \geq 1$, and as 1, if $n = 0$.

We will solely concentrate on the case when $U = U(P, Q)$ is a nondegenerate fundamental Lucas sequence, i.e., a second-order linear recurrent sequence with $U_0 = 0$, $U_1 = 1$ and $U_n \neq 0$ for $n \geq 2$ which satisfies

$$U_{n+2} = PU_{n+1} - QU_n, \text{ for all } n \geq 0,$$

where $P$ and $Q$ are nonzero integers. Those generalized binomial coefficients are referred to as Lucasnomials. Thus, the sequence $I = (I_n)_{n\geq 0}$ of natural numbers is the particular fundamental Lucas sequence $U(2, 1)$ which yields the ordinary binomial coefficients.

We know $n + 1$ divides $\binom{2n}{n}$ for all $n \geq 1$. If $k$ is an integer not 1, then there are infinitely many $n \geq 1$ for which $n + k$ does not divide $\binom{2n}{n}$. As it happens this Catalan phenomenon remains true, or nearly so, for Lucasnomials. That is, for $\gcd(P, Q) = 1$,

$$\frac{1}{U_{n+k}} \binom{2n}{n}_U$$

is an integer for all $n \geq 1$ iff $k = 1$, or $k = 2$ and $U = U(1, 2)$. We will review further extensions of these results, in particular with respect to Lucasnomial Fuss-Catalan numbers and go in detail over theorems surrounding this phenomenon. Open problems will be outlined.

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**Barry Balof**, Whitman College, Walla Walla, WA, USA
*Selfish Sets, Posets, Tilings and Bijections*

A subset of the integers $\{1, 2, \ldots, n\}$ is *selfish* if it contains its own cardinality as an element. Those sets for which the minimal element is the cardinality (referred to by Grimaldi as *extraordinary* sets) are enumerated by the Fibonacci Numbers. In a 2013 paper, Grimaldi and Rickert introduced a partial order on these extraordinary sets. In this talk, we will establish natural bijections between the subsets and domino-square tilings to give a new interpretation to some combinatorial identities.

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**Bob Bastasz**, Missoula, MT, USA
*Digital Loop Systems*

A digital loop system $S[m, l]$ is a set of periodic sequences based on a $l$-order linear recurrence in a finite field $\mathbb{F}_m$. Each sequence, called a loop, is expressed as a Lyndon word consisting of the digits in its least period and can be uniquely specified by a minimal element, which is a $l$-tuple pre-necklace. The periods of all distinct loops in a system sum to $m^l$. For example, the Fibonacci sequence (mod 10), with a period of 60, is one of six digital loops contained the system $S[10, 2]$, whose periods sum to $10^2$.

A basic property of a digital loop system is the number of distinct loop periods, $c$. Of particular interest are systems in which $m$ is a prime and $c$ is two or three. If $S[m, l]$ has $c$=2, it is proposed that $S[m^i, l]$ has $c = i + 1$ where $i$ is a positive integer. Cases where the same period can be found for loops in $S[m, l]$ and $S[m^2, l]$ will be discussed.

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**Arthur T. Benjamin**, Harvey Mudd College, Claremont, CA, USA
*Some Bingo Paradoxes*

In the game of Bingo, when many cards are in play, it is much more probable that the winning card is horizontal than vertical. We will explore this and other paradoxes. Fibonacci numbers and q-binomial coefficients make a brief appearance.

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**Bruce M. Boman**, University of Delaware, Newark, DE, USA
*Geometric Branching Patterns Based on the p-Fibonacci Numbers: Self-similarity Across Different Degrees of Branching and Multiple Dimensions*

Branching patterns occur throughout nature and are often described by the Fibonacci numbers. While the regularity of these branching patterns in biology can be described by the Fibonacci numbers, the branches (leaves, petals, offshoots, limbs, etc) are often variegated (size, color, shape, etc). To begin to understand how these patterns arise, we considered different branching patterns based on the $p$-Fibonacci numbers. In our model, different branch patterns were created based on a specific number of decreasing-sized branches that arise from a main branch (termed the degree of branching). It was assumed that the ratio between the sizes of pairs of consecutive branches (ordered by size) equals the ratio of the largest branch size to the sum of the largest and smallest branch sizes. Generation of these branching structures illustrates that pattern self-similarities occur across different degrees of branching and multiple dimensions. Conclusion: studying geometric branching patterns based on the $p$-Fibonacci numbers begins to show how the regularity in branching patterns might occur in biology.

(Joint work with Gilberto Schleiniger).

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**Scott Cameron**, Dalhousie University, Halifax, NS, Canada
*A Linear Algebra Problem Related to Legendre Polynomials*

I introduce a problem which piqued my interest, namely a question asked in the context of simple linear algebra, and then generalize this problem to investigate further properties. This leads to a study of families of polynomial coefficients for kernels of shifted Legendre polynomials, and the properties which they have. It turns out that there is a general formula for the generating function of each of these families.

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**Marc Chamberland**, Grinnell College, Grinnell, IA, USA
*Arctan Formulas and $\pi$*

There are many interesting formulas connected to the arctan function. For centuries, Machin-like formulas, such as

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)$$

were the main technique used for calculating Pi. Starting with a geometric motivation, we build several new arctan formulas, for example,

$$\pi = \arctan\left(a\sqrt{\frac{a+b+c}{abc}}\right) + \arctan\left(b\sqrt{\frac{a+b+c}{abc}}\right) + \arctan\left(c\sqrt{\frac{a+b+c}{abc}}\right)$$

when $a, b, c > 0$. (Joint work with Eugene Herman.)

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**Meliza Contreras González**, Universidad Autónoma de Puebla, Puebla, Mexico

*Counting Independent Sets on Bipolygonal Graphs*

We consider the sequence $\beta_{s,k} = F_s \cdot F_{k-s}$ for $k > 0, 1 \leq s \leq k-1$, introduced in [1], that is formed by the product of two Fibonacci numbers with complementary indexes. The values of this sequence allow us to compute the number of independent sets on bipolygonal graphs, which are graphs formed by two polygons $C_i$ and $C_j$ joined by an edge $e = \{x, y\}$, with $x \in V(C_i)$ and $y \in V(C_j)$. We denote this class of graphs as $H_{i,j}$. In particualr, when the polygons $C_i$ and $C_j$ are hexagons, then $H_{i,j}$ is the primitive graph used to form chains of polyphenylene compounds [2].

We apply the edge division rule to decompose $H_{i,j}$ and to use the values in the sequence $\beta_{s,k}$ for computing the number of independent sets of $H_{i,j}$, denoted as $i(H_{i,j})$. In fact, $i(H_{i,j}) = F_{i+1} \cdot F_{j+1} + F_{i+1} \cdot F_{j-1} + F_{i-1} \cdot F_{j+1}$. Fixing $k \geq 6$, and $k = i + j$, we consider the different subgraphs formed by the variations: $3 \leq i, j \leq (k-3)$. We analyze all possible size combinations for $C_i$ and $C_j$, fixing $i + j$ as a constant $k$.

In addition, the way to compute $H_{i,j}$ allow us to determine extremal topologies for $i(H_{i,j})$. The extremal values are identified when the greatest variation (entropy) between the sizes of the polygons $C_i$ and $C_j$ is achieved. The minimum value corresponds to $|C_j| - |C_i| = 6$, and the maximum value is given when $|C_j| - |C_i| = 4$.

(Joint work with Guillermo De Ita Luna and Pedro Bello López.)

**References**

[1] G. De Ita, J. R. Marcial, J. A. Hernández, R. M. Valdovino, Extending Extremal Polygonal Arrays for the Merrifield-Simmons Index, *Lecture Notes in Computer Science*, **10267** (2017), 22–31.

[2] Došlić T., Litz M. S., Matchings and Independent Sets in Polyphenylene Chains, *MATCH Commun. Math. Comput. Chem.*, **67** (2012), 313–330.

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**Curtis Cooper**, Univ. of Central Missouri, Warrensburg, MO, USA

*Some Generalized High Order Fibonacci Identities*

The Gelin-Cesáro identity states that for integers $n \geq 2$,

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = -1,$$

where $F_n$ denotes the $n$th Fibonacci number. Horadam generalized the Fibonacci sequence by defining the sequence $W_n$ where $W_0 = a$, $W_1 = b$, and $W_n = pW_{n-1} - qW_{n-2}$ for $n \geq 2$ and $a$, $b$, $p$ and $q$ are integers and $q \neq 0$. Using this sequence, Melham and Shannon generalized the Gelin-Cesáro identity by proving that for integers $n \geq 2$,

$$W_{n-2}W_{n-1}W_{n+1}W_{n+2} - W_n^4 = eq^{n-2}(p^2+q)W_n^2 + e^2q^{2n-3}p^2,$$

where $e = pab - qa^2 - b^2$. We will discover and prove some similar generalized high order Fibonacci identities.

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**Steven Edwards**, Kennesaw State University, Marietta, GA, USA
*Generalizations of Delannoy Numbers*

The Delannoy numbers $D(m,n)$ count the number of lattice paths from $(0,0)$ to $(m,n)$, where the allowable steps are up, right, and diagonal. The Delannoyi numbers satisfy the recursion $D(m,n) = D(m,n-1) + D(m-1,n) + D(m-1,n-1)$. By restricting the number of diagonal steps allowed, we construct collections of generalized Delannoy numbers. The generalized Delannoy numbers satisfy the same recursion as the Delannoy numbers. There are many relations relating these numbers, and these relations, in turn, produce binomial identities which generalize known identities. Some of these identities provide insights into intrinsic properties of Pascal's triangle.

(Joint work with William Griffiths.)

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**Larry Ericksen**, Millville, NJ, USA
*Properties of Polynomials that Encode Representations*

We present hyperbinary properties of two-variable Stern polynomials, with extension to hyper b-ary representations of integers. Continued fractions are also constructed from polynomial analogues of Lucas sequences.

(Joint work with Karl Dilcher.)

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**Dale Gerdemann**
*Images from Zeckendorf and Other Numerical Representations*

Images, many of them fractal, can be generated from generalizations of numerical representations, primarily Golden Ratio Base, Zeckendorf and a variant of Zeckendorf which uses negatively indexed Fibonacci numbers (due to Martin Bunder). Bit sequences can be extracted from these representations that can be used to guide a walk in the plane, using color coding to represent the number of times each lattice point is encountered. Several algorithms can be used to convert the binary bit-pattern sequence into directional information, but all rely on the use of a division test using a prespecified divisor. The test may be on a count of the total number of steps already taken in the walk or it may be on a more restricted count of just the steps corresponding to a 1-bit in the bit sequence. The idea is extended to Lucas numbers of the first kind: $U_0 = 0$, $U_1 = 1$, $U_n = sU_{n-1} + tU_{n-2}$. A variety of examples for various $s, t$ and divisor can be seen at `https://bit.ly/2KcKSL7`. Fractal images exhibit symmetry at specific points in their construction. Based on experimentation, these points may correspond to Lucas numbers of the first or second kind. Possibly path lengths required to construct an image of some sort could be used to provide a counting interpretation of other combinatorial integer sequences. To test this idea, images are constructed for OEIS A181926 (diagonal sums of Fibonomial triangle).

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**Heiko Harborth**, TU Braunschweig, Braunschweig, Germany
*A Conjecture for Pascal's Triangle*

For a prime number $p$ consider Pascal's triangle reduced modulo $p$. Let $a_i(n)$ count the number of residues $i$ in row $n$. If the linear combinations

$$c_0a_0(n) + c_1a_1(n) + ... + c_{p-1}a_{p-1}(n) = 0$$

are fulfilled then it is conjectured by H.-D. Gronau and M. Krueppel that $c_i = 0$ for $0 \le i \le p-1$. Partial results are presented.

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**Russell Jay Hendel**, Towson University, Towson, MD, USA
*Proof and Formulation of a Tagiuri-Generating-Method Conjecture*

The Tagiuri Generating Method (TGM) generates one-parameter, infinite, families, of Fibonacci identities. To describe, $I(q)$, the $q$-th member of a family of identities, we need functions $s_p(q), s_n(q), s(q)$, and $m(q)$. Let $P = \prod_{j=1}^{m(q)} F_{n+a_j}$ with the $a_j, 1 \le j \le m(q)$, parameters. TGM *starts* with an identity of the form $(s_p(q) - s_n(q))P = s_p(q)P - s_n(q)P$. TGM then requires replacement of one product $F_{n+a_k}F_{n+a_l}$ in each of the $s(q) = s_p(q) + s_n(q)$ summands on the right-hand side of the *start* identity with the corresponding right-hand side of the basic Tagirui identity $F_{n+a_k}F_{n+a_l} = F_nF_{n+a_k+a_l} + (-1)^nF_{a_k}F_{a_l}$. Since the start and Tagiuri identities are true, $I(q)$ is also true. In practice, we map $\{a_j, 1 \le j \le m(q)\}$ to a subset of the integers symmetric around 0 (and excluding 0 if $m(q)$ is even). Particular examples of TGM families have been explored in FQ articles and conferences (Fibonacci (CAEN), West Coast Number Theory, MASON II). Main results are typically expressed as statements about the $I(q)$-*index histograms*, $H_q = \{(x, c_q(x)) : x \in \mathbb{Z}\}$, where for each integer $x$, $c_q(x)$

counts the number of occurrences of $F_{n+x}$ in $I(q)$. In this talk, we prove, *Under mild restrictions,* $\#\{c_q(x) : (x, c_q(x)) \in H_q, x, q \in \mathbb{Z}, q \geq 1.\} \leq c$, *that is, for each* $q \geq 1$, *the number (cardinality, #) of distinct index-counts in* $I(q)$ *is bounded above by a very small computable constant,* $c$, *independent of* $q$. The theorem is proven by presenting a single proof unifying all previous examples. The presentation closes by reviewing the history of Fibonacci-Lucas identities and showing a very recent trend to studying families of identities instead of individual identities or proof methods.

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**Burghard Herrmann**, Köln, Germany
*How Integer Sequences Find Their Way Into Areas Outside "Pure Mathematics"*

Integers are considered on the surface of a cylinder as in the model of the pineapple in Coxeter [Introduction to Geometry, Chapter 11.5 Phyllotaxis (literally "leaf arrangement")]. The fractional part $n\Phi$ determines the angular position of the $n$th leaf measured in turn, where $\Phi$ denotes the golden ratio. A positive integer $n$ is called a "front number" if $n\Phi < 1/4$ or $n\Phi > 3/4$, otherwise, $n$ is called a "back number".

The sequence of front numbers (http://oeis.org/A295085) is related with some sequences of Kimberling: the sequence of front numbers is the intertwining of A190249 and A190251 and the sequence of back numbers corresponds to A190250.

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**Burghard Herrmann**, Köln, Germany
*Visibility in a Pure Model of Golden Spiral Phyllotaxis*

As a contribution to our understanding of lattices the talk summarizes the paper "Visibility in a pure model of golden spiral phyllotaxis". It is published in the current issue of MATHEMATICAL BIOSCIENCES (Share Link: https://authors.elsevier.com/a/1X9t75pvHBD-A).

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**Orli Herscovici**, University of Haifa, Haifa, Israel
*New Degenerated Bernoulli and Euler Polynomials Arising From Non-Classical Umbral Calculus*

We introduce new generalizations of the Bernoulli and Euler polynomials based on the degenerate exponential function and concepts of the Umbral Calculus associated with it. We present generalizations of some familiar identities and connection between these kinds of Bernoulli and Euler polynomials which we have established in our preliminary work.

(Joint work with Toufik Mansour.)

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**Lin Jiu**, Dalhousie University, Halifax, NS, Canada
*Bessel Random Walks and Identities for Higher-Order Bernoulli and Euler Polynomials*

We consider the study of random walks as a technique to obtain non-elementary identities for higher-order Euler and Bernoulli polynomials. In the case of a one-dimensional linear reflected Brownian motion, considering the successive hitting times of uniformly distributed levels in $[0, 1]$ yields non-trivial expressions for higher-order Euler polynomials. These results are also interpreted as a stochastic sum decomposition due to Klebanov. Analogous results in the case of a 3-dimensional Bessel process yield non-elementary expressions about higher-order Bernoulli polynomials.
(Joint work with Christophe Vignat.)

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**Bir Kafle**, Purdue University Northwest, Westville/Hammond, IN, USA
*Pell Numbers of the Form* $2^a + 3^b + 5^c$

The Pell sequence $(P_n)_{n\geq 0}$, Pell-Lucas sequence $(Q_n)_{n\geq 0}$ and the associated Pell sequence $(q_n)_{n\geq 0}$ are defined by the same binary recurrences

$$P_{n+1} = 2P_n + P_{n-1}, \quad Q_{n+1} = 2Q_n + Q_{n-1} \quad \text{and} \quad q_{n+1} = 2q_n + q_{n-1},$$

with the initial terms $P_0 = 0, P_1 = 1$, $Q_0 = Q_1 = 2$ and $q_0 = q_1 = 1$, respectively. The problem of finding Fibonacci, Lucas, or Pell numbers of a particular form has a very rich history. In this talk, we look into $P_n$, $Q_n$ and $q_n$ as the sum of the three perfect powers of some prescribed distinct bases. In particular, we determine all the solutions of the Diophantine equations

$$P_n = 2^a + 3^b + 5^c, \quad Q_n = 2^a + 3^b + 5^c \quad \text{and} \quad q_n = 2^a + 3^b + 5^c$$

in positive integers $(n, a, b, c)$, with some restrictions. Our methods involve the linear forms in logarithms of algebraic numbers.
(Joint work with F. Luca and A. Togbé.)

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**Clark Kimberling**, University of Evansville, Evansville, IN, USA
*Linear Complementary Equations and Systems*

After a brief history of complementary equations, a definition is given for linear complementary equation, with particular attention to examples typified by $a_n = a_{n-1} + a_{n-2} + b_n$, where $(b_n)$ is the complement of $(a_n)$ in the set $N$ of positive integers, and $a_n/a_{n-1} \to (1+\sqrt{5})/2$. Also introduced are systems of equations whose solutions are sequences that partition $N$. An example is the system defined recursively by $a_n =$ least new $k$, $b_n =$ least new $k$, and $c_n = a_n + b_n$, where "least new $k$", also known as "mex", is the least integer in $N$ not yet placed. The sequence $(c_n)$ in this example is the anti-Fibonacci sequence, A075326 in the Online Encyclopedia of Integer Sequences.

(Joint work with Peter J. C. Moses.)

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**Karyn McLellan**, Mount Saint Vincent University, Halifax, NS, Canada
*A Problem on Generating Sets Containing Fibonacci Numbers*

At the Sixteenth International Conference on Fibonacci Numbers and Their Applications the following problem was posed:

Let $S$ be the set generated by these rules: Let $1 \in S$ and if $x \in S$, then $2x \in S$ and $1 - x \in S$, so that $S$ grows in generations: $G_1 = \{1\}, G_2 = \{0, 2\}, G_3 = \{-1, 4\}, \ldots$
Prove or disprove that each generation contains at least one Fibonacci number or its negative.

We will show that every integer $k$ can be found in some $G_i$ and will disprove the above statement by finding an expression for the generation index $i$ for any given $k$. We will use a variety of recurrence sequences including the dragon curve sequence, properties of binary numbers, and a computer calculation to find numerous counterexamples.

(Joint work with Danielle Cox)

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**Steven J. Miller**, Williams College, Williamstown, MA, USA
*From Monovariants to Zeckendorf Decompositions and Games*

Zeckendorf's Theorem states that every positive integer has a unique decomposition as a sum of non-adjacent Fibonacci numbers; this has been generalized to many other recurrences. We show by looking at appropriate monovariants that these decompositions have the fewest summands possible. We use this perspective to analyze a new two-player game on Fibonacci decompositions, and provide a non-constructive proof that Player Two always has a winning strategy. As time permits we will discuss generalizations and open problems.

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**Antara Mukherjee**, The Citadel, Charleston, SC, USA
*The Geometric Interpretation of Some Fibonacci Identities in the Hosoya Triangle*

The Hosoya triangle is a triangular array (like the Pascal triangle) where the e ntries are products of Fibonacci numbers. The symmetry present in the Hosoya tri angle helps us explore several patterns and find new identities. In this talk we give a geometric interpretation -using the Hosoya triangle- of s everal Fibonacci identities that are well known algebraically. For example, we d iscuss geometric proofs of Cassini, Catalan, and Johnson identities. We also ext end some properties from Pascal triangle to the Hosoya triangle. For instance, w e generalize the hockey stick property, the T-stick identities –that were origin ally given in terms of binomial coefficients– to identities for Fibonacci numbers.

(Joint work with R. Flórez and R. Higuita.)

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**Kouichi Nakagawa**, Saitama University, Saitama, Japan
*Exact Periodicity of Generalized Fibonacci and Tribonacci Sequence*

Let $\{G_n(a,b)\}$ be the generalized Fibonacci sequence, where $G_0 = a$, $G_1 = b$, and $G_n = G_{n-1} + G_{n-2}$, $n \geq 2$, and let $\{T_n(a,b,c)\}$ be the generalized Tribonacci sequence, where $T_0 = a$, $T_1 = b$, $T_2 = c$ and $T_n = T_{n-1}+T_{n-2}+T_{n-3}$, $n \geq 3$. (Thus $G_n(0,1)$ is the $n$th Fibonacci number, $G_n(2,1)$ is the $n$th Lucas number, $T_n(0,0,1)$ is the $n$th Tribonacci number (or $n$th Fibonacci 3-step number) and $T_n(3,1,3)$ is the $n$th Lucas 3-step number).

D. D. Wall showed that the generalized Fibonacci sequence is simply periodic when taken modulo $m$. (For example, in the case of the original Fibonacci sequence the length of period (mod 10) is 60, and (mod 100) it is 300, and so on.) C. C. Yalavigi showed that the generalized Tribonacci sequence mod $m$ is simply periodic as well. (For example, in the case of the original Tribonacci sequence the length of period (mod 10) is 124 and (mod 100) is 1240, and so on.) However, the simply periodic sequences do not necessarily have the smallest period. (For example, the generalized Tribonacci sequences (mod 10), $(a,b,c) = (3,1,3), (1,7,9), \ldots$ have a period of 31, $(a,b,c) = (0,1,0), (3,2,1), \ldots$ have a period of 62 and $(a,b,c) = (0,0,1), (2,3,5), \ldots$ have a period of 124.) Hence we compute the periods (mod $10^d$) up to $d = 4$ for all generalized Fibonacci sequences and up to $d = 3$ for all Tribonacci sequences by experimental mathematics and analyze the relationships between periodic groups observed.

(Joint work with Rurika Sudo.)

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**Sam Northshield**, SUNY-Plattsburgh, Plattsburgh, NY, USA

*Re³counting the Rationals*

In 1999, Neil Calkin and Herbert Wilf wrote their charming "Recounting the rationals" which gave an explicit bijection between the positive integers and the positive rationals: namely, $n \mapsto a_{n+1}/a_n$ where $a_n$ is defined by $a_1 = 1, a_{2n} = a_n$, and $a_{2n+1} = a_{n+1} + a_n$. Alternatively, $a_{n+1}$ is the number of hyperbinary representations of $n$ or, in still another way,

$$a_{n+1} = a_n + a_{n-1} - 2\left(a_{n-1} \mod a_n\right).$$

Expressing this in another way, for $f(x) := 1+1/x-2\{1/x\}$ where $\{x\}$ denotes the fractional part of $x$, the sequence $1, f(1), f(f(1)), f(f(f(1))), \ldots$ is a list of all of the positive rationals.

We will discuss the facts that iterates of $2+2/x-4\{1/x\}$ starting at 2 also cover the positive rationals as do the iterates of $3+3/x-6\{1/x\}$ starting at 3. That is, the iterates of $cf(x)$, starting at $c$, cover the positive rationals when $c = 1, 2, 3$. Surprisingly, $c = 1, 2, 3$ are the only numbers for which this is true.

I'll sketch some of the proofs; they involve, among other things, "negative" continued fractions, Chebyshev polynomials, Euler's totient function, arrangements of circles, and the generalized Stern sequences

$$x_{n+1} = \sqrt{c} \cdot x_n + x_{n-1} - 2\left(x_{n-1} \mod \left(\sqrt{c} \cdot x_n\right)\right).$$

I will also discuss some remarkable properties of these latter sequences.

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**Prapanpong Pongsriiam**, Silpakorn University, Faculty of Science, Nakhon Pathom, Thailand

*Fibonacci and Lucas Numbers Which Have Exactly Three Prime Factors and Some Unique Properties of $F_{18}$ and $L_{18}$*

Let $F_n$ and $L_n$ be the $n$th Fibonacci and Lucas numbers, respectively. Let $\omega(n)$ be the number of prime factors of $n$, $d(n)$ the number of positive divisors of $n$, $A(n)$ the least positive reduced residue system modulo $n$, and $\ell(n)$ the length of longest arithmetic progressions contained in $A(n)$. In the occasion of attending the 18th Fibonacci Conference, we will show some results concerning $\omega(F_n)$, $\omega(L_n)$, $d(F_n)$, and $d(L_n)$ which reveal a unique property of $F_{18}$ and $L_{18}$. We also find the solutions to the equation $\ell(n) = 18$ and show a connection between them and $F_{18}$. Some examples and numerical data will also be presented.

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**Murat Şahin**, Ankara University, Tandogan Ankara, Turkey
*Fibonacci Numbers and Core Partitions*

There are many studies about core partitions and such partitions are closely related to posets, cranks, Raney numbers, Catalan numbers, Fibonacci numbers etc. In particular, the number of $(s, s+1)$-core partitions with distinct parts equals the Fibonacci numbers. We study $(s, s+1)$-core partitions with $d$-distinct parts and provide some results on the number and the largest size of such partitions. Also, we propose a conjecture about $(s, s+r)$-core partitions with $d$-distinct parts for $1 \leq r \leq d$.

(Joint work with Elif Tan).

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**J. C. Saunders**, University of Waterloo, Waterloo ON, Canada
*On $(a, b)$ Pairs in Random Fibonacci Sequences*

We examine the random Fibonacci tree, which is an infinite binary tree with non-negative integers at each node. The root consists of the number 1 with a single child, also the number 1. We define the tree recursively in the following way: if $x$ is the parent of $y$, then $y$ has two children, namely $|x - y|$ and $x + y$. This tree was studied by Benoit Rittaud who proved that any pair of integers $a, b$ that are coprime occur as a parent-child pair infinitely often. We extend his results by determining the probability that a random infinite walk in this tree contains exactly one pair $(1, 1)$, that being at the root of the tree. Also, we give tight upper and lowerbounds on the number of occurrences of any specific coprime pair $(a, b)$ at any given fixed depth in the tree.

(Joint work with Kevin Hare.)

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**Susanna Spektor**, Brock University, St. Catharines, ON, Canada
*On a $\psi_1$-Norm Estimate of Sums of Dependent Random Variables Using Simple Random Walks on Graphs*

We obtained a $\psi_1$ estimate for the sum of Rademacher random variables under condition that they are dependent.

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**Paul K. Stockmeyer**, The College of William & Mary, Williamsburg, VA, USA
*Discovering Fibonacci Numbers, Fibonacci Words, and a Fibonacci Fractal in the Tower of Hanoi*

The Tower of Hanoi puzzle, with three pegs and $n$ graduated discs, was invented by Edouard Lucas in 1883, writing under the name of Professor N. Claus. A simple question about relative distances between various regular states of this puzzle has lead to the discovery of a new occurrence of Fibonacci numbers, a new illustration of the finite Fibonacci words, and a fractal of Hausdorff dimension $\log_2(\phi)$, where $\phi$ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

(joint work with Andreas M. Hinz, LMU München, Germany.)

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**Elif Tan**, Ankara University, Ankara, Turkey
*A Note on Conditional Divisibility Sequences*

A sequence of rational integers $\{a_n\}$ is said to be a *divisibility sequence (DS)* if $m \mid n$ whenever $a_m \mid a_n$ and it is said to be a *strong divisibility sequence (SDS)* if $\gcd(a_m, a_n) = a_{\gcd(m,n)}$. These sequences are of particular interest because of their remarkable factorization properties and usage in applications, such as factorization problem, primality testing, etc. The best known examples are Fibonacci sequence, Lucas sequence, Lehmer sequence, Vandermonde sequences, resultant sequences and their divisors, elliptic divisibility sequences, etc.

In this talk, we consider the conditional recurrence sequence $\{q_n\}$ is one in which the recurrence satisfied by $q_n$ depends on the residue of $n$ modulo some integer $r \geq 2$. If the conditional sequence $\{q_n\}$ is also a divisibility or strong divisibility sequence, we call it as a *conditional divisibility* or *conditional strong divisibility sequence*. We investigate and find some families of the conditional divisibility and the conditional strong divisibility sequences.

(Joint work with Murat Şahin.)

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**Thotsaporn 'Aek' Thanatipanonda**, Mahidol University International College, Nakornphathom, Thailand
*Statistics of Domino Tilings on a Rectangular Board*

It is well known the Fibonacci sequence, $F_n$, is the number of ways to cover a 2-by-$(n-1)$ board using only the horizontal$(H)$ or vertical$(V)$ 2-by-1 dominos. It is natural to generalize this idea to a rectangular $m$-by-$n$ board where $m$ is a fixed number and $n$ is symbolic. We can try harder and compute the mixed moment $S[V^a H^b]$ for fixed non-negative integers $a, b$ but general $m, n$. After all these moments are computed, we will gain an information of the distribution of these statistics as well. Note that the Fibonacci numbers and generalization are the cases where $a = b = 0$ i.e. the zero moment.

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**Christophe Vignat**, Université d'Orsay, Orsay, France
*Finite Generating Functions for the Sum-of-Digits Sequence*

I will show some results about finite generating functions associated with the sequence $\{s_b(n)\}$, where $s_b(n)$ is the sum of the digits of the representation in base $b$ of the integer $n$. This sequence has been studied, for example, by J.-P. Allouche and J. Shallit – see their book "Automatic Sequences, Theory, Applications, Generalizations". Thanks to a general identity that relates the sequence $\{s_b(n)\}$ to the finite difference operator, we obtain, for example, an explicit expression for a Hurwitz-type generating function related to this sequence. Our generalizations also include links to some Lambert series and to infinite products related to the sequence $s_b(n)$.

(Joint work with T. Wakhare.)

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**Tanay Wakhare**, University of Maryland, College Park, MD, USA
*Structural Identities for Multiple Zeta Values*

medskip We revisit some results by Borwein et al. about Multiple Zeta Values and show that they can be extended to an arbitrary, possibly finite, sequence of numbers. Specializing these sequences as the zeros of special functions gives us some new results about Bessel and Airy Multiple Zeta values. In the Bessel case, specializing the argument to $\nu = 1/2$ allows us to recover the classical results by Borwein.

(Joint work with C. Vignat.)

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**William Webb**, Washington State University, Pullman, WA, USA
*What Makes A "Nice" Identity?*

There are probably thousands of known identities involving recurrence sequences. We will suggest one way to judge whether an identity is particularly simple, or nice, or maybe unexpected. We begin with a reminder that often the easiest way to prove many of the basic properties of recurrence sequences, including proving identities, is viewing recurrences as elements of vector spaces. By looking at the dimensions of these vector spaces we can show why some identities are rather ordinary and others much more unexpected. We end by showing how these techniques can be used to prove a conjecture by R. S. Melham.

**William Webb**, Washington State University, Pullman, WA, USA
*Proving Identities in Arbitrary Fields*

Most of the known identities involve the Fibonacci numbers or possibly other recurrence sequences in the ring of integers. However, the natural place to study them is in the complex field, since we often need to express a recurrence as a generalized power sum in terms of powers of the roots of the associated characteristic polynomial. Since we need roots of polynomials it is easier to work over an algebraically closed field. It is often the case that an identity (or other result) is first proved for the Fibonacci numbers, them maybe for the Lucas numbers, Pell numbers, other second order integer sequences, Fibonacci polynomials etc. However, it is possible to prove all of these cases, as well as for recurrence sequences in much more general algebraically closed fields, all at once. Since the results are much more general, the proofs may be more tedious. Many proof techniques can be used, but some, such combinatorial proofs which usually involve quantities counted by integers, may not be applicable. We give several examples of such general identities, such as:

If a second order recurrence sequence satisfies the recurrence $u_{n+2} = au_{n+1} + bu_n$, then

$$\sum_{j=0}^{n} u_j = \frac{bu_n + u_{n+1} - 1}{a + b - 1}.$$

This is of course not a new identity, but we note that it is true regardless of whether the parameters $a$ and $b$ and hence the recurrence sequence itself are integers, polynomials, power series, $p$-adic numbers etc.

(Joint work with Nathan Hamlin.)

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**Paul Young**, College of Charleston, Charleston, SC, USA
*Global Series for Zeta Functions*

We give two general classes of everywhere-convergent series for Barnes generalization of Hurwitz zeta functions, which involve Bernoulli polynomials of the second kind and weighted Stirling numbers. These series are also valid $p$-adically, and yield several identities and series for zeta values and Stieltjes constants which are valid in both real and $p$-adic senses.

**Paul Young**, College of Charleston, Charleston, SC, USA
*The Power of 2 Dividing a Generalized Fibonacci Number*

Let $T_n$ denote the generalized Fibonacci number of order $k$ defined by the recurrence $T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}$ for $n \geq k$, with initial conditions $T_0 = 0$ and $T_i = 1$ for $1 \leq i < k$. Motivated by some recent conjectures of Lengyel and Marques, we establish the 2-adic valuation of $T_n$, settling one conjecture affirmatively and one negatively. We discuss the computational issues that arise and applications to Diophantine equations involving $(T_n)$.

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**Osman Yürekli**, Ithaca College, Ithaca, NY, USA
*A Pascal-Like Triangle From a Special Function*

This presentation is devoted to a new Pascal-like triangle appearing unexpectedl y from a sequence of polynomials obtained from the derivatives of the special function Dawson's integral which is defined by the integral

$$\mathrm{daw}(x) = \int_0^x \exp(y^2 - x^2)\, dy.$$

We investigate the properties of the Pascal-like triangle and its applications. In addition, we discuss a generalization for the triangle and Dawson's integral. It is also possible to obtain new Fibonacci-like sequences from the triangle and its generalizations.